CHAPTER 8

Inversion

Out of nothing I have created a strange new universe. — János Bolyai

In this chapter we discuss the method of inversion in the plane. This technique is useful for turning circles into lines and for handling tangent figures.

8.1 Circles are Lines

A **cline** (or **generalized circle**) refers to either a circle or a line. Throughout the chapter, we use "circle" and "line" to refer to the ordinary shapes, and "cline" when we wish to refer to both.

The idea is to view every line as a circle with infinite radius. We add a special point P_∞ to the plane, which every ordinary line passes through (and no circle passes through). This is called the **point at infinity**. Therefore, every choice of three distinct points determines a unique cline—three ordinary points determine a circle, while two ordinary points plus the point at infinity determine a line.

With this said, we can now define an inversion. Let ω be a circle with center O and radius R. We say an **inversion** about ω is a map (that is, a transformation) which does the following.

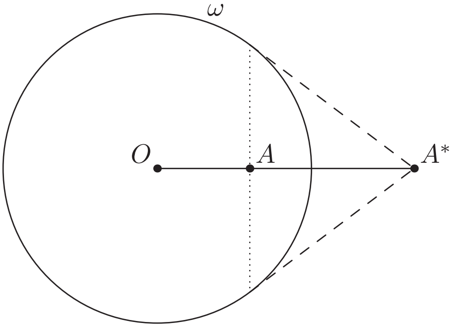

Figure 8.1A. A^* is the image of the point A when we take an inversion about ω.

- The center O of the circle is sent to P_∞.
- The point P_∞ is sent to O.

- For any other point A, we send A to the point A^* lying on ray OA such that $OA \cdot OA^* = r^2$.

Try to apply the third rule to $A = O$ and $A = P_\infty$, and the motivation for the first two rules becomes much clearer. The way to remember it is "$\frac{r^2}{0} = \infty$" and "$\frac{r^2}{\infty} = 0$".

At first, this rule seems arbitrary and contrived. What good could it do? First, we make a few simple observations.

1. A point A lies on ω if and only if $A = A^*$. In other words, the points of ω are fixed.
2. Inversion swaps pairs of points. In other words, the inverse of A^* is A itself. In still other words, $(A^*)^* = A$.

We can also find a geometric interpretation for this mapping, which provides an important setting in which inverses arise naturally.

Lemma 8.1 (Inversion and Tangents). *Let A be a point inside ω, other than O, and A^* be its inverse. Then the tangents from A^* to ω are collinear with A.*

This configuration is shown in Figure 8.1A. It is a simple exercise in similar triangles: just check that $OA \cdot OA^* = r^2$.

This is all fine and well, but it does not provide any clue why we should care about inversion. Inversion is not very interesting if we only look at one point at a time—how about two points A and B?

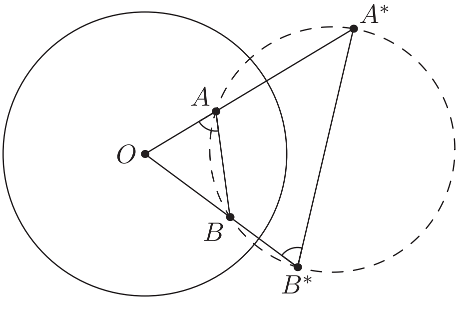

Figure 8.1B. Inversion preserves angles, kind of.

This situation is shown in Figure 8.1B. Now we have some more structure. Because $OA \cdot OA^* = OB \cdot OB^* = r^2$, by power of a point we see that quadrilateral ABB^*A^* is cyclic. Hence we obtain the following theorem.

Theorem 8.2 (Inversion and Angles). *If A^* and B^* are the inverses of A and B under inversion centered at O, then $\measuredangle OAB = -\measuredangle OB^*A^*$.*

Unfortunately, this does not generalize nicely* to arbitrary angles, as the theorem only handles angles with one vertex at O.

It is worth remarking how unimportant the particular value of r has been so far. Indeed, we see that often the radius is ignored altogether; in this case, we refer to this as **inversion**

* The correct generalization is to define an angle between two clines to be the angle formed by the tangents at an intersection point. This happens to be preserved under inversion. However, this is in general not as useful.

around $\boldsymbol{P}$, meaning that we invert with respect to a circle centered at P with any positive radius. (After all, scaling r is equivalent to just applying a homothety with ratio r^2.)

Problem for this Section

Problem 8.3. If z is a nonzero complex number, show that the inverse of z with respect to the unit circle is $(\overline{z})^{-1}$.

8.2 Where Do Clines Go?

So far we have derived only a few very basic properties of inversion, nothing that would suggest it could be a viable method of attack for a problem. The results of this section will change that.

Rather than looking at just one or two points, we consider entire clines. The simplest example is a just a line through O.

Proposition 8.4. *A line passing through O inverts to itself.*

By this we mean that if we take each point on a line ℓ (including O and P_∞) and invert it, then look at the resulting locus of points, we get ℓ back again. The proof is clear.

What about a line not passing through O? Surprisingly, it is a circle! See Figure 8.2A

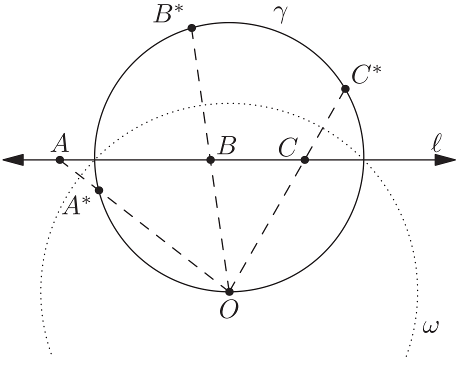

Figure 8.2A. A line inverts to a circle through O, and vice versa.

Proposition 8.5. *The inverse of a line ℓ not passing through O is a circle γ passing through O. Furthermore, the line through O perpendicular to ℓ passes through the center of γ.*

Proof. Let ℓ^* be the inverse of our line. Because P_∞ lies on ℓ, we must have O on ℓ^*. We show ℓ^* is a circle.

Let A, B, C be any three points on ℓ. It suffices to show that O, A^*, B^*, C^* are concyclic. This is easy enough. Because they are collinear, $\measuredangle OAB = \measuredangle OAC$. Using Theorem 8.2, $\measuredangle OB^*A^* = \measuredangle OC^*A^*$, as desired. Since any four points on ℓ^* are concyclic, that implies ℓ^* is just a circle.

It remains to show that ℓ is perpendicular to the line passing through the centers of ω (the circle we are inverting about) and γ. This is not hard to see in the picture. For a proof, let X be the point on ℓ closest to O (so $\overline{OX} \perp \ell$). Then X^* is the point on γ farthest from O, so that $\overline{OX^*}$ is a diameter of γ. Since O, X, X^* are collinear by definition, this implies the result. □

In a completely analogous fashion one can derive the converse—the image of a circle passing through O is a line. Also, notice how the points on ω are fixed during the whole transformation.

This begs the question—what happens to the other circles? It turns out that these circles also invert to circles. Our proof here is of a different style than the previous one (although the previous proof can be rewritten to look more like this one). Refer to Figure 8.2B.

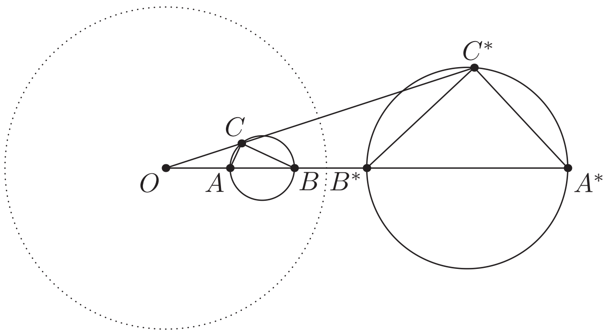

Figure 8.2B. A circle inverts to another circle.

Proposition 8.6. *Let γ be a circle not passing through O. Then γ^* is also a circle and does not contain O.*

Proof. Because neither O nor P_∞ is on γ, the inverse γ^* cannot contain these points either. Now, let $\overline{AB}$ be a diameter of γ with O on line AB (and $A, B \neq O$). It suffices to prove that γ^* is a circle with diameter $\overline{A^*B^*}$.

Consider any point C on γ. Observe that

$$90^\circ = \measuredangle BCA = -\measuredangle OCB + \measuredangle OCA.$$

By Theorem 8.2, we see that $-\measuredangle OCA = \measuredangle OA^*C^*$ and $-\measuredangle OCB = \measuredangle OB^*C^*$. Hence, a quick angle chase gives

$$90^\circ = \measuredangle OB^*C^* - \measuredangle OA^*C^* = \measuredangle A^*B^*C^* - \measuredangle B^*A^*C^* = -\measuredangle B^*C^*A^*$$

and hence C^* lies on the circle with diameter $\overline{A^*B^*}$. By similar work, any point on γ^* has inverse lying on γ, and we are done. □

It is worth noting that the centers of these circles are also collinear. (However, keep in mind that the centers of the circle do not map to each other!)

We can summarize our findings in the following lemma.

Theorem 8.7 (Images of Clines). *A cline inverts to a cline. Specifically, in an inversion through a circle with center O,*

(a) A line through O inverts to itself.
(b) A circle through O inverts to a line (not through O), and vice versa. The diameter of this circle containing O is perpendicular to the line.
(c) A circle not through O inverts to another circle not through O. The centers of these circles are collinear with O.

We promised that inversion gives the power to turn circles into lines. This is a result of (b)—if we invert through a point with many circles, then all those circles become lines.

Finally, one important remark. Tangent clines (that is, clines which intersect exactly once, including at P_∞ in the case of two lines) remain tangent under inversion. This has the power to send tangent circles to parallel lines—we simply invert around the point at which they are internally or externally tangent.

Problems for this Section

Problem 8.8. In Figure 8.2C, sketch the inverse of the five solid clines (two lines and three circles) about the dotted circle ω. **Hint**: 279

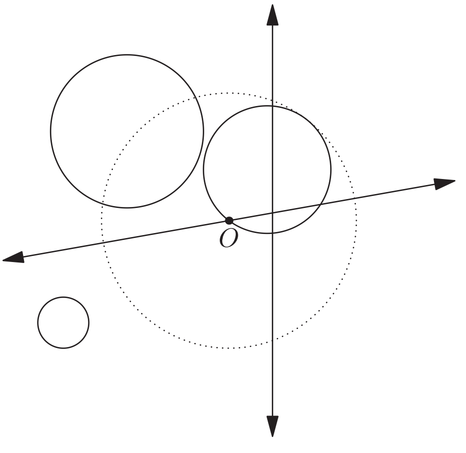

Figure 8.2C. Practice inverting.

Lemma 8.9 (Inverting an Orthocenter). *Let ABC be a triangle with orthocenter H and altitudes $\overline{AD}$, $\overline{BE}$, $\overline{CF}$. Perform an inversion around C with radius $\sqrt{CH \cdot CF}$. Where do the six points each go?* **Hint**: 257

Lemma 8.10 (Inverting a Circumcenter). *Let ABC be a triangle with circumcenter O. Invert around C with radius 1. What is the relation between O^*, C, A^*, and B^*?* **Hint**: 252

Lemma 8.11 (Inverting the Incircle). *Let ABC be a triangle with circumcircle Γ and contact triangle DEF. Consider an inversion with respect to the incircle of triangle ABC. Show that Γ is sent to the nine-point circle of triangle DEF.* **Hint**: 560

8.3 An Example from the USAMO

An example at this point would likely be illuminating. We revisit a problem first given in Chapter 3.

Example 8.12 (USAMO 1993/2). Let $ABCD$ be a quadrilateral whose diagonals $\overline{AC}$ and $\overline{BD}$ are perpendicular and intersect at E. Prove that the reflections of E across $\overline{AB}$, $\overline{BC}$, $\overline{CD}$, $\overline{DA}$ are concyclic.

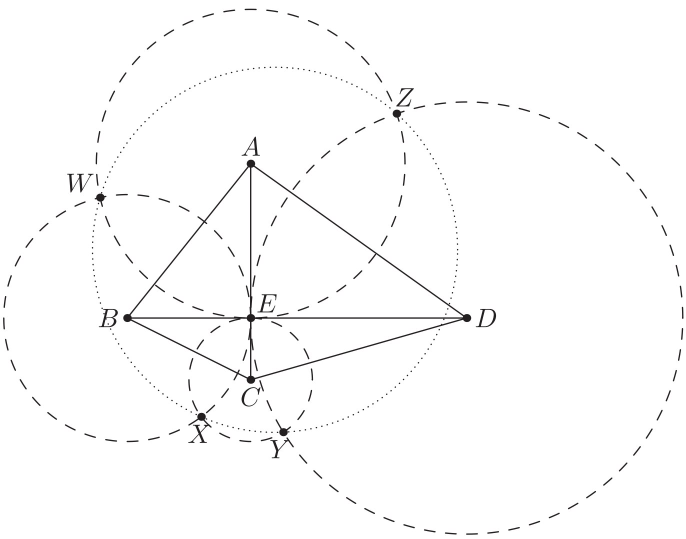

Figure 8.3A. Adding in some circles.

Let the reflections respectively be W, X, Y, Z.

At first, this problem seems a strange candidate for inversion. Indeed, there are no circles. Nevertheless, upon thinking about the reflection condition one might notice

$$AW = AE = AZ$$

which motivates us to construct a circle ω_A centered at A passing through all three points. If we define ω_B, ω_C, and ω_D similarly, suddenly we no longer have to worry about reflections. W is the just the second intersection of ω_A and ω_B, and so on.

Let us rephrase this problem in steps now.

1. Let $ABCD$ be a quadrilateral with perpendicular diagonals that meet at E.
2. Let ω_A be a circle centered at A through E.
3. Define ω_B, ω_C, ω_D similarly.
4. Let W be the intersection of ω_A and ω_B other than E.
5. Define X, Y, Z similarly.
6. Prove that $WXYZ$ is concyclic.