A geometric approach to continued fractions

September 25, 2019

1 Continued fractions [1]

In the following we will use the notation,

$$[b : a_0, a_1, \ldots, a_n] = b + \cfrac{1}{a_0 + \cfrac{1}{a_1 + \cfrac{1}{\ddots \, \cfrac{}{a_{n-1} + \cfrac{1}{a_n}}}}}.$$

Exercise 1. *Evaluate* $[0 : 1, 2, 3, 4, 5]$.

We denote the convergents of a continued fraction by, for $k > 0$,

$$[0 : a_0, a_1, \ldots, a_k] = \frac{p_k}{q_k}$$

Exercise 2. *Show that the convergents satisfy,*

$$p_n = a_{n-1}p_{n-1} + p_{n-2} \quad \textit{and} \quad q_n = a_{n-1}q_{n-1} + q_{n-2}$$

for all $n > 1$*, where we assume* $p_0 = 0$ *and* $q_0 = p_1 = q_1 = 1$*.*

Exercise 3. *What is* $\frac{p_n}{q_n} - \frac{p_{n-1}}{q_{n-1}}$ *?*

Exercise 4. *Find the simple continued fractions expansion for* $81/35$ *and* $277/101$ *and use these to find integers* m *and* n *for which* $81m + 35n = 1$ *and* $277m + 101n = 1$*.*

Exercise 5. *Find the continued fraction expansion for* $\frac{10!+1}{11!+1}$ *and* $\frac{3^7-1}{3^8-1}$*.*

Exercise 6. *Evaluate in terms of some square root (not a decimal expression) the number to which converges,*

$$[5, 2, 3, 2, 1, 3, 2, 1, 3, 2, 1, 3, 2, 1, \ldots].$$

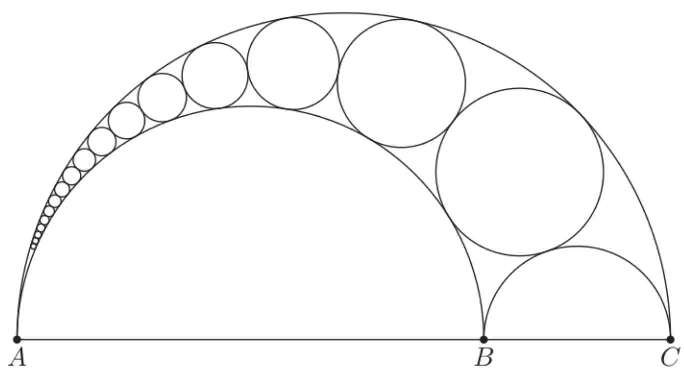
Exercise 7. *Find the continued fraction expansion for the golden ration* $\phi = \frac{1+\sqrt{5}}{2}$*, then for* $\sqrt{20}$*.*

[1] Most of the exercises in this section come from a sheet written by Ted Alper for Standford Math Circle.

2 Plane inversion

There are several good exercises in the handout on plane inversion. Here is a little extra,

Exercise 8. *Let A, B, C be three collinear points (in that order) and construct three semicircles $\Gamma_{AC}, \Gamma_{AB}, \omega_0$, on the same side of AC, with diameters AC, AB, BC, respectively. For each positive integer k, let ω_k be the circle tangent to Γ_{AC} and Γ_{AB} as well as ω_{k-1}. Let n be a positive integer. Prove that the distance from the center of ω_n to AC is n times its diameter.*



3 Farey tessellation and Ford circles.

Consider an *ideal triangle* T defined in the upper-half plane by the two vertical lines passing through 0 and 1, and the half-circle or diameter $\overline{01}$. We denote by $\iota_{0\infty}, \iota_{1\infty}, \iota 01$ the plane inversion the corresponding plane inversions. The Farey tessellation is the *orbit*, or in other words, the images of T by any composition of these three maps. These images are called *tiles* of the tessellation.

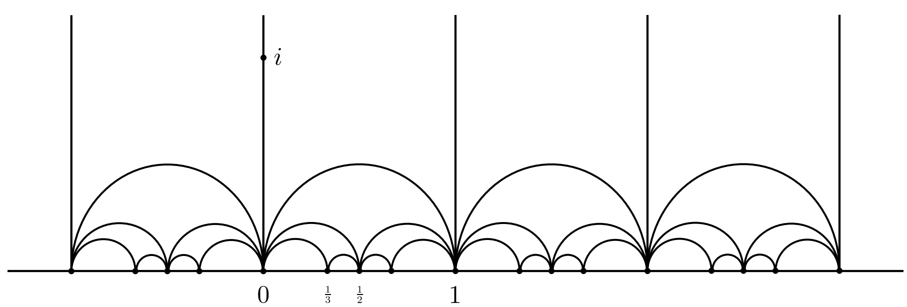


Figure 1: Farey Tessellation.

Exercise 9. *Show that the images by any composition of the three inversions are not overlapping.*

Problem 1 (C. Series' theorem)**.** *Let x be a real number represented as a point on the horizontal line, and consider the circle passing through the point i and x. Each time it crosses a triangle, starting with T, it has to go left of right, we denote by $w = LLLRRLR\dots$ the word describing this sequence if left and right. Factoring the same letters, we write*

$$w = L^{a_0} R^{a_1} L^{a_2} \dots,$$

show that

$$x = [a_0; a_1, a_2, \dots].$$

Exercise 10. *Deduce from this theorem that the Farey tessellation covers the whole upper half-plane, and that the tessellation meets the horizontal line at rational numbers.*

We define the *wrong addition* of fractions by the formula

$$\frac{p}{q} \oplus \frac{p'}{q'} = \frac{p+q}{p'+q'}.$$

Exercise 11. *Consider a tile of the Farey tessellation which meets the horizontal line at x, y, z (in increasing order), prove that $y = x \oplus z$.*

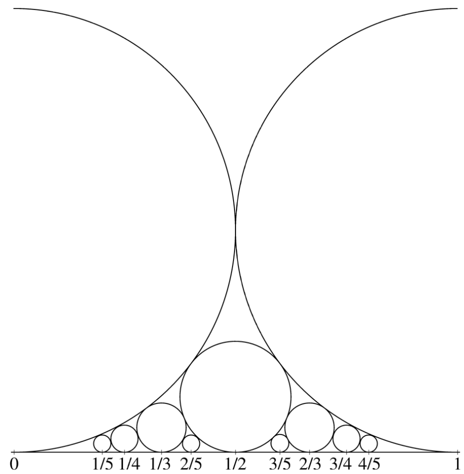
Problem 2 (Farey's theorem)**.** *Show that the set of tiles that edges on the horizontal line with a denominator smaller than n,*

$$I_n = \{\frac{p}{q} \mid 0 \leq p \leq q,\ 0 < q \leq n\}$$

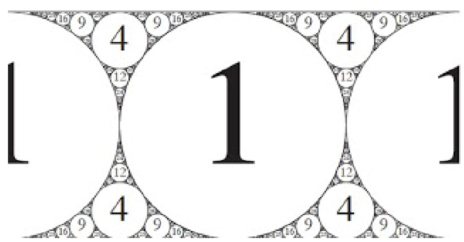
does not meets the horizontal line in other points. Consider now the elements of I_n order in increasing order $u_0, u_1, \dots, u_m$.

Prove that for all $n > 0$, $u_n = u_{n-1} \oplus u_{n+1}$

The Farey tessellation is closely related to Ford sphere packing.



All the circles have radius $\frac{1}{n^2}$ for all $n \geq 1$.



I will probably not have the time to develop, but you can look them up on the internet. Both of these objects are closely related to hyperbolic geometry.