Intermediate II Berkeley Math Circle: Graph Coloring Problems

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1 Graphs and graph coloring

A number of different problems in mathematics can be reduced to the problem of coloring the nodes of a graph, and we'll spend the session today exploring such problems.

A **graph** consists of a set of **nodes** or **vertices**, connected by a number of **edges**. We will require a graph to satisfy a few rules:

(i) Each edge must connect two distinct vertices; we don't allow an edge to connect a vertex back to itself.

(ii) Each pair of distinct vertices can have *at most* one edge between them; we don't allow multiple edges between the same pair of vertices.

(iii) We will assume there are a finite number of vertices, and hence an finite number of edges. (While there are interesting problems concerning infinite graphs, we won't talk about these today!)

(iv) Finally, we'll require a graph to be **connected**: from any vertex, it is possible to reach any other vertex by following some path along edges.

The number of edges connected to a vertex is called its **degree**. By the connectivity requirement, all vertices have degree at least 1.

Note that it doesn't matter how we draw the graph. The edges need not be straight lines, and we can allow the edges to cross one another (for now). As long as two graphs have the same number of vertices which are connected in the same way by edges, we regard the two graphs as being **the same**. Put another way, if we take a drawing of a graph and move the vertices around in some way, dragging the edges along, the resulting drawing still represents the same graph as the original.

Figure 1 shows several examples of graphs. Which ones are the same?

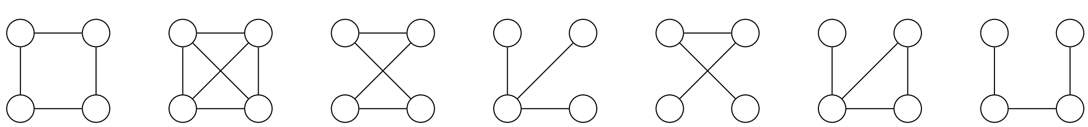

Figure 1: Some graphs with 4 vertices.

The problems we'll consider today are all about **coloring** graphs. The idea is to assign a color to each vertex (or we can use numbers or letters, which is especially useful if you don't have a bunch of colored pencils or pens!) without breaking the following rule:

Two vertices which are connected by an edge cannot have the same color.

The idea is to find the fewest number of colors possible to color a given graph. A graph which can be colored using k colors is called **k-colorable**, and the minimum number k such that a graph is k colorable is called its **chromatic number**.

Problem 1. Determine the chromatic numbers of the graphs in Figure 1.

Problem 2. Find examples of graphs with chromatic numbers 4, 5, and 6.

2 Complete graphs and trees

We next introduce some more terminology and some special classes of graphs.

A graph is called **complete** if every vertex is connected to every other one. Note that any two complete graphs with equal numbers of vertices are the same.

A **cycle** inside a graph is a path along a sequence of distinct edges (no edge can be traversed twice) which starts and ends at the same vertex. A graph which contains no cycles is called a **tree**. Why do you think they are so named?

Problem 3. What is the chromatic number of the complete graph on n vertices?

Problem 4. What is the chromatic number of an arbitrary tree? (Try some examples first and then see if you can formulate a general answer).

Problem 5. Suppose a graph can be built up starting from a single vertex by repeatedly adding new vertices which connect to *at most* 2 of the existing ones. What is the chromatic number of such a graph?

Problem 6.

(a) What is the relationship between the number of vertices and the number of edges for a complete graph?

(b) What is the relationship between the number of vertices and the number of edges for a tree?

Problem 7. Show that for any graph, the sum of the degrees of all its vertices equals twice the number of edges.

3 Art Gallery Problem

Here is a fun problem that is not at first sight related to graph coloring. Say we have an art gallery whose floor plan is a simple polygon[1]. We have a bunch of expensive art hung on the walls and want to make sure the art is guarded at all times. Suppose each guard is represented by a point, and that each guard can see in all directions until their line of sight is blocked by a wall. We can station the guards anywhere, but assume that they will not move.

[1] Recall that a simple polygon is a plane region bounded by a finite sequence of noncrossing straight line segments, which is connected and has no holes. It need not be convex (and the problem is not very interesting if it is!)

Problem 8 (Art Gallery Problem)**.** For an aribtrary art gallery with n vertices, what is the minimum number of guards required so that every point in the gallery is visible to at least one guard?

4 Planar graphs and Euler's theorem

A graph is **planar** if it can be drawn on a plane in such a way that no edges cross one another. For the rest of the session, we will focus only on planar graphs.

One way to get a planar graph is to start with a **map** of 2D regions separated by borders, such as countries, states, counties, etc. If we turn each region into a vertex and connect them by a graph if the regions share a nonzero length of border (no corners!), then we get a planar graph. Such a graph associated to the map of the continental United States appears in Figure 2.

A famous problem in mathematics, to which we will soon return, is to find the minimum number of colors needed to color every possible 2D map, real or imagined; such maps can be pretty wild! The map coloring problem is completely equivalent to the problem of coloring planar graphs.

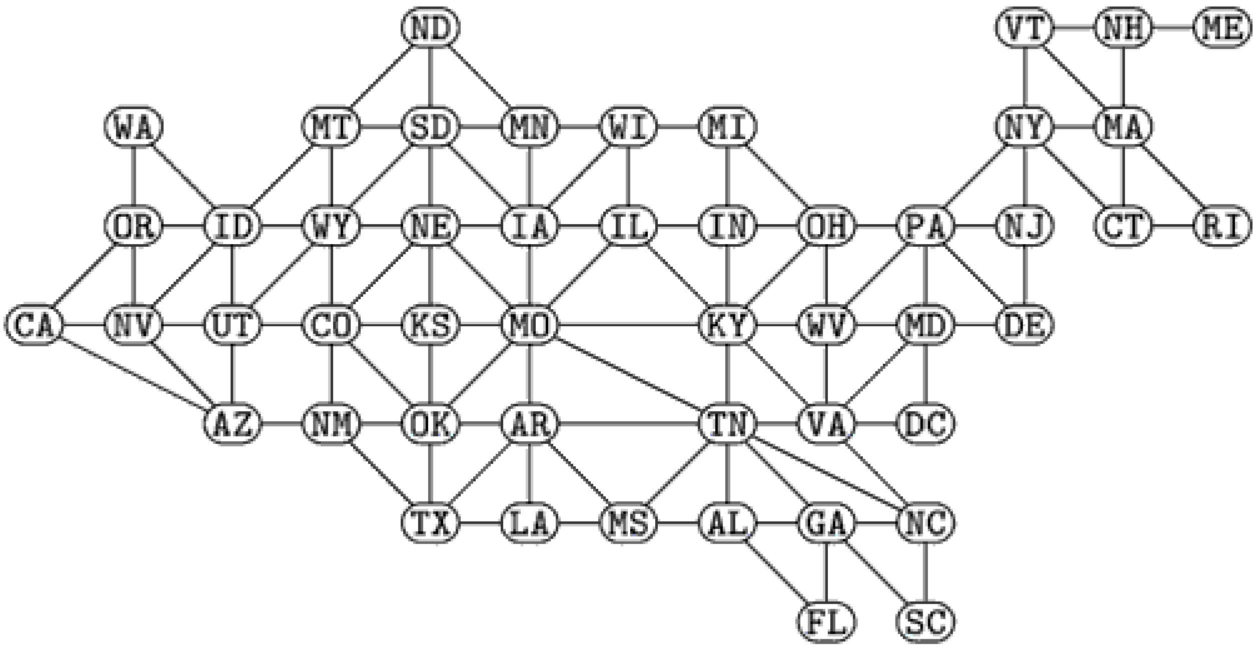

Figure 2: The continental US as a graph.

Problem 9. Which complete graphs (all vertices connected to all other vertices) are planar?

Problem 10. A graph is called **regular** if all of its vertices have the same degree (i.e., same number of edges).

(a) Find an example of a regular planar graph all of whose vertices have degree 4.

(b) How about degree 5?

(c) Can you find an example where each degree is 6? (Read ahead if you get stuck!) Remember that edges can't cross one another in a planar graph.

Problem 11. Find a coloring of the USA map graph in Figure 2 using a minimal number of colors. How many colors did you need?

In addition to the vertices and edges of a planar graph, we can think about the 2D regions bounded by the edges, which we will call its **faces**. Along with the obvious faces "inside" the graph, we we also consider the region "outside" the graph as a single face. A famous theorem by Leonhard Euler gives a relation between the number of vertices, edges, and faces which holds for any planar graph.

Theorem 4.1 (Euler's theorem)**.** *If a planar graph has V vertices, E edges, and F faces (including the outside face), then*

$$V - E + F = 2. \tag{1}$$

Just like for Problem 7, we can try to determine a relation between E and F by counting the number of edges adjacent to a face and vice versa, but instead of an equality, we will get an inequality.

Lemma 4.2. *For every planar graph with at least 3 vertices, three times the number of faces is less than or equal to twice the number of edges:*

$$3F \leq 2E. \tag{2}$$

Corollary 4.3. *In every planar graph, there is at least one vertex of degree* 5 *or less.*

5 Five Color Theorem

A famous theorem says that every planar graph (equivalently, every 2D map) can be colored using four colors or less. The proof of this Four Color Theorem is incredibly difficult (the original version was over 400 pages!), and was the first theorem to use computers in an essential way, to check a large number of special cases that would be essentially impossible to check by hand. For this reason it was slow to gain acceptance among some mathematicians.

In contrast to the Four Color Theorem, the slightly weaker Five Color Theorem is much easier to prove!

Theorem 5.1 (Five Color Theorem)**.** *Every planar graph is 5-colorable.*

Proof idea for Theorem 4.1: Suppose the graph contains a cycle. If we delete an edge along this cycle, then the graph remains connected (why?), and while the number of edges decreases by one, so does the number of faces (why?) so the number $V - E + F$ stays the same. We can keep on deleting edges along cycles until there are no more cycles left, in which case what remains is a tree, but for a tree $V - E = 1$ and $F = 1$ (why?) so the equation holds. □

Proof idea for Lemma 4.2: Each face is bounded by at least 3 edges (why?), but if we start enumerating edges this way for every face, many edges will be counted multiple times. Since each edge is adjacent to at most 2 faces (why?), each edge is counted at most twice this way, so we get $\frac{3}{2}F \leq E$ as claimed. □

Proof idea for Corollary 4.3: If we plug in (2) into (1), we get

$$2 = V - E + F \leq V - E + \frac{2}{3}E = V - \frac{1}{3}E,$$

or

$$E \leq 3V - 6, \tag{3}$$

which gives an upper bound on the number of edges of a planar graph in terms of the number of vertices (provided $V \geq 3$).

If all vertices had degree 6 or more, then from Problem 7 we would have

$$2E \geq 6V \quad \text{or} \quad E \geq 3V,$$

but combining this with (3), we get the impossible to satisfy inequality

$$3V \leq E \leq 3V - 6,$$

from which we conclude that it is not possible for every vertex to have degree 6 or more. In other words, at least one vertex has degree 5 or less. □

Proof idea for Five Color Theorem: If a graph has 5 or fewer vertices, then it is obviously 5-colorable, so we have proved the result in this case. The strategy is to use this to show that all planar graphs with 6 vertices are 5-colorable, and then use this to show that all planar graphs with 7 vertices are 5-colorable, and so on. This is known as *proof by induction*. Rather than do each step explicitly (which would take forever!), we give a general argument. Thus we assume that we are in the nth step of this process: we have already proved that planar graphs with $n-1$ vertices are 5-colorable, and we need now to show that an arbitrary planar graph with n vertices is 5-colorable.

By Corollary 4.3, some vertex of our graph has degree at most 5. Call this vertex v. If we remove v and its adjacent edges, we are left with a planar graph with $n-1$ vertices, which we can color with 5 colors by the inductive assumption. If v had degree 4 or less, or if we did not use all 5 colors on the vertices adjacent to v when coloring the smaller graph, then it is easy to pick a color for v and hence color our n vertex graph consistently.

So we may assume we are in the worst case: v has 5 adjacent vertices, which are colored Red, Orange, Yellow, Green, and Blue in order. Starting with the Red and Yellow vertices, let us plot out the subgraphs consisting of vertices colored with these two colors only. If the Red and Yellow vertices are not connected by such a subgraph, then we could switch Yellow and Red in one of the two subgraphs without violating the color rule, and this would free up a color (Red or Yellow) to use for v. Thus let us assume we are again in the worst case, which is that the Red and Yellow vertices are connected by some path which alternates between Red and Yellow vertices.

Now play the same game again with the Orange and Green vertices. It *cannot* be the case that these two vertices are connected by a Orange-Green path, since this path would have to cross the Red-Yellow path and this is not possible. It follows that in one of the Orange-Green subgraphs, we can swap Orange and Green and free up a color for v.

In all cases we have shown that we can go from a 5-coloring of the graph with v removed, and then color v in a consistent way so that our original graph is 5-colored, and since the graph was arbitrary, we have proved that all graphs with n vertices can be 5-colored. $\square$