HYPERBOLIC GEOMETRY

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Incidence plane is a set of *points* and a collection of subsets called *lines* satisfying the following properties

(1) For any two distinct points A and B there exists a unique line containing A and B.
(2) Every line contains at least two points.
(3) There are three points not on the same line.

1. List all finite incidence planes with 3,4 and 5 points,

2. In a finite incidence plane the number of lines is not less than the number of points.

3. Two lines are called parallel if they do not have common points. An incidence plane is called affine if for every line l and a point A not on l there exists and a unique line m containing A and parallel to l. Show that

(a) Every line in an affine plane has the same number of points.

(b) If an affine plane is finite then the number of points equals n^2 where n is the number of points on a line.

(c) Show that there exist affine planes with p^2 points for every prime number p.

Euclidean plane and rigid motions. Let Π denote the Euclidean plane. A transformation of Π is called a *rigid motion* if it preserves distances and therefore angles.

4. A composition of two rigid motions is a rigid motion. For every rigid motion F there exists a rigid motion G such that $F \circ G$ and $G \circ F$ is the identity.

5. If a rigid motion preserves three noncollinear points, then it is the identity.

6. If a rigid motion preserves two distinct points A and B, then it preserves all points on the line through A and B. If this motion is not identity then it is the reflection in the line AB.

7. If a rigid motion has exactly one fixed point then it is a rotation.

8. If a rigid motion does not have fixed points, then it is either a parallel translation or a glide reflection (i.e. a composition of a reflection in a line and a translation along that line).

9. A composition of two rotations is either a rotation or a parallel translation.

10. Every rigid motion can be written as a composition of one, two or three reflections.

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11. Identify the Euclidean plane with the set of complex numbers. Then a rigid motion F is given by the formula $F(z) = az + b$ or $F(z) = a\bar{z} + b$ for some complex numbers a and b, $|a| = 1$. If we drop the condition $|a| = 1$, then F will preserve angles and map a triangle to a similar triangle.

Given a circle C with center at a point O and radius r, *inversion* or *reflection* in C is a transformation which maps every point $A \neq O$ to the point A' lying on the ray OA such that $|OA||OA'| = r^2$.

12. An inversion maps a circle not containing O to another circle and a circle containing O to a line.

13. Given four distinct points in a plane, we define the *cross-ratio*

$$(AB, PQ) = \frac{|AP|}{|AQ|} : \frac{|BP|}{|BQ|}.$$

Check that a circular inversion preserves cross-ratio, i.e. $(AB, PQ) = (A'B', P'Q')$ where X' denotes the image of X under inversion.

14. A *linear fractional transformation* of a complex plane is defined by the formula

$$F(z) = \frac{az + b}{cz + d}$$

for some complex numbers a, b, c, d such that $ad - bc \neq 0$.

(a) A composition of two linear fractional transformations is again a linear fractional transformation.

(b) A composition of a circular inversion and complex conjugation is a linear fractional transformation.

15. Let $J(z) = \frac{1}{z}$. Any linear fractional transformation is either linear or a composition $L \circ J \circ L'$ for some linear transformations L and L'.

16. A linear fractional transformation preserves cross-ratio.

17. A linear fractional transformation maps a circle (line) to a circle or line and preserves angles between lines and circles.

18. Let $\mathcal{H}$ denote the upper half of the complex plane. Check that a linear fractional transformation maps $\mathcal{H}$ to itself if and only if $F(z) = \frac{az+b}{cz+d}$ for real a, b, c, d such that $ad - bc > 0$.

A *Lobachevsky line* (L-line) is a subset of $\mathcal{H}$ consisting of points lying on the same vertical line or on the same circle with center at the real line. The **Lobachevsky plane** is $\mathcal{H}$ with the set of L-lines. The real line is called the *horizon* of the Lobachevsky plane.

19. The Lobachevsky plane is an incidence plane.

We can define angles between L-lines (since they are lines or circles in the usual sense). Then the linear fractional transformations defined in Problem 18 preserve angles.

The next step is to define the distance on the Lobachevsky plane. Given two distinct points A and B in $\mathcal{H}$, the L-line passing through A and B meets the horizon at two points (if L-line is a vertical line we think of one of these two points as the infinite point). We denote these two points P and Q so that B lies between P and A and set $\mu(A,B) = (AB, PQ)$. If $A = B$ we define $\mu(A,B) = 1$ The function μ is called the *multiplicative distance*. Note that $\mu(A,B) \geq 1$ and $\mu(A,B) = 1$ if and only if $A = B$.

20. Check that if A, B, C lie on the same L-line and B is between A and C, then $\mu(A,C) = \mu(A,B)\mu(B,C)$. This explains the name "multiplicative distance". In fact, if A, B, C are non-collinear we have the triangle inequality

$$\mu(A,C) \geq \mu(A,B)\mu(B,C).$$

21. If F is a rigid motion of the Lobachevsky plane, then $F(z) = \frac{az+b}{cz+d}$ for real a, b, c, d such that $ad - bc > 0$ or $F(z) = \frac{a\bar{z}+b}{c\bar{z}+d}$ for real a, b, c, d such that $ad - bc < 0$.

22. Many theorems of Euclidean geometry hold in the Lobachevsky plane:

(a) A triangle is isosceles if and only if its base angles are equal.

(b) In a triangle an exterior angle is large that a non-adjacent interior angle.

(c) In any triangle the larger side supports the larger angle.

(d) Given a line l and a point A there exists a unique line passing through A and perpendicular to l.

23. Let us call two triangles ABC and $A'B'C'$ *congruent* if the corresponding angles and corresponding sides are equal. Check that two triangles are congruent if and only if there exists a rigid motion which maps ABC to $A'B'C'$.

24. Certain properties of triangles remain the same as in Euclidean geometry.

(a) If ABC and $A'B'C'$ have their respective sides equal then they are congruent.

(b) If $\mu(AB) = \mu(A'B')$, $\mu(AC) = \mu(A', C')$ and $\angle BAC = \angle B'A'C'$ then ABC and $A'B'C'$ are congruent.

(c) If $\mu(AB) = \mu(A'B')$, $\angle BAC = \angle B'A'C'$ and $\angle ABC = \angle A'B'C'$ then ABC and $A'B'C'$ are congruent.

But some things are completely different.

25. In the Lobachevsky plane the sum of angles in any triangle is less than $180°$.

$\delta(ABC) = 180° - (\angle A + \angle B + \angle C)$ is called the *defect* of a triangle ABC.

26. (a) Let D lie on the side AB of a triangle ABC then

$$\delta(ABC) = \delta(ADC) + \delta(BDC).$$

(b) If a triangle DEF lies inside ABC then $\delta(DEF) < \delta(ABC)$.

27. There are no similar triangles. If ABC and $A'B'C'$ have their respective angles equal then they are congruent.

28. For every angle $\alpha < 60°$ there exists an equilateral triangle with angle α.

29. Let ABC be an equilateral triangle with angle $45°$. Find $\mu(AB)$.

30. Show that $\mathcal{H}$ can be covered by equilateral triangles with angle $45°$ without overlapping.