

**Cuts and flows in a graph:** Let  $G$  be a directed graph with set of vertices  $V$  and edges  $E$ , where each edge  $u \rightarrow v$  in  $E$  is assigned a *capacity*  $c_{uv}$ . Two vertices  $s$  and  $t$  are designated *start* and *end*.

A *flow* from  $s$  to  $t$  is an assignment of a number  $f_{uv}$  to every edge  $u \rightarrow v$  satisfying two conditions:

- (a) For every edge  $u \rightarrow v$ ,  $0 \leq f_{uv} \leq c_{uv}$ .
- (b) For every vertex  $v$  except from  $s$  and  $t$ , “flow in = flow out”:

$$\sum_{u \rightarrow v} f_{uv} - \sum_{v \rightarrow u} f_{vu} = 0. \quad (1)$$

The *value* of a flow is the quantity (1) for the end vertex  $t$  (or its negative for the start vertex  $s$ ).

A *cut* from  $s$  to  $t$  is a subset of  $E$  such that every path from  $s$  to  $t$  uses one of the edges in  $E$ . Its *value* is the total capacity of all edges in the cut.

**Flow-cut duality theorem:** The maximum value of a flow in  $G$  is equal to the minimum value of a cut in  $G$ .

**Matchings and covers:** Now let  $G$  be a bipartite graph, with set of vertices  $V = L \sqcup R$  and edges only from  $L$  to  $R$ . A *matching* is a set of edges of  $G$  sharing no common vertices. A matching is *perfect* if every vertex is contained in one of the edges. A *cover* is a set of vertices such that every edge contains one of them.

**König’s theorem:** The maximum size of a matching in  $G$  is equal to the minimum size of a cover of  $G$ .

**Hall’s “marriage” theorem:** Suppose  $|L| = |R|$ . A perfect matching exists in  $G$  if and only if for every subset  $S \subseteq L$ , the number of vertices in  $R$  joined to at least one vertex in  $S$  has size at least  $|S|$ .

**Problems:**

- (1) Certain people of gender  $A$  like certain people of gender  $B$ . Show the following are equivalent: (a) for every set of  $n$   $A$ -gendered people, at least  $n$   $B$ -gendered people are liked by at least one of them (for every  $n$ ); (b) for every set of  $n$   $B$ -gendered people, at least  $n$   $A$ -gendered people like at least one of them. (There are the same number of people of each gender.)
- (2) (“Polygamous marriage theorem”) Suppose  $|R| = k|L|$ . Every vertex in  $L$  is to be matched with exactly  $k$  vertices in  $R$ , and every vertex in  $R$  with exactly 1 vertex in  $L$ . This is possible if and only if every subset  $S \subseteq L$  is joined to at least  $k|S|$  vertices in  $R$ .
- (3) A complete set of the Encyclopedia of Mathematics has 99 volumes. There are 99 mathematicians in Berkeley, and each of them owns four volumes of the Encyclopedia. Together they own four complete sets. Show that there is a way for each mathematician to donate one book to the library so that the library receives a complete set.
- (4) A round-robin tournament of  $2n$  teams lasted for  $2n - 1$  days, as follows. On each day, every team played one game against another team, with one team winning and one team losing in each of the  $n$  games. Over the course of the tournament, each team played every other team exactly once. Can one necessarily choose one winning team from each day without choosing any team more than once?
- (5) A sheet of paper of area 100 is colored into 100 regions, each of area 1. Then the sheet is turned over and colored into 10 regions, each of area 10. Prove that one can make 100 holes in the paper so that all 110 regions have been punctured.
- (6) 32 rooks are placed on a chessboard so that there are 4 rooks in each row and in each column. Can one always find a set of 8 rooks that pairwise do not attack each other? (Bonus: Replace 4 by any number  $n$  from 1 to 8 and 32 by  $8n$ .)