BMC Spring 2019 - Beginners

Board Tiling and Chocolate Sharing with a hint of the Fibonacci sequence - Part III

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I. Overarching question

Given a $2 \times n$ bar of chocolate, how many ways can we break it into n pieces of 2×1 (domino style) to share to n friends?

II. Warm-Ups

We will write the Fibonacci sequence in a strange way. We will use powers of a dummy variable x to keep track of the order. That is, each number in the sequence will be attached to a power of x, where the power tells us which term it is, as follow: Instead of writing

$$1, 1, 2, 3, 5, ..., \text{ we write}$$

$$F = 1x^1 + 1x^2 + 2x^3 + 3x^4 + 5x^5 ...$$

.

Then, what does it mean if you see the term $89x^{11}$ in F?

III. Technical Preparation

We want to derive a formula for the number in front of the general power x^n in the expanded form of F. That number, called the coefficient, will be the *nth* Fibonacci number.

1. Let $\phi_1 = \frac{\sqrt{5}-1}{2}, \phi_2 = \frac{\sqrt{5}+1}{-2}$. Do the following calculations:

   a) $\phi_1 - \phi_2$

   b) $\phi_1 \times \phi_2$

c) $-\phi_1^2 - \phi_1 + 1$

2. Perform the following calculation

$$\left(\frac{1}{8-x} - \frac{1}{13-x}\right)\frac{1}{8-13}$$

3. Prove (or believe)

$$\sum_{n=0}^{\infty} \frac{1}{2^n} = \frac{1}{2^0} + \frac{1}{2} + \frac{1}{2^2} + \frac{1}{2^3} + \frac{1}{2^4} + ... = 2$$

4. Prove (or believe)

$$\sum_{n=0}^{\infty} x^n = \frac{1}{1-x}.$$

Does it match with the number we found earlier?

IV. Putting facts together

Calculate:
$F - xF - x^2F =$

Then, we have $F = \frac{x}{1-x-x^2}$.
From here, we perform a lot of mathemagic to get to the final formula:

$$a_n = \frac{1}{\sqrt{5}}\Big[\Big(\frac{1+\sqrt{5}}{2}\Big)^n - \Big(\frac{1-\sqrt{5}}{2}\Big)^n\Big].$$

You should check the first few numbers to see if it works!