Strolling with Euler: Extradimensional Exploration

Let's start with a simple question. How do we find the sum of the interior angles of a convex polygon?

- List as many methods as you can.
- Think about: how can you adapt your method to concave polygons?

Formula for the interior angles of a polygon: _________________ .

Part 2 – Platonic Solids Again

Definition: A platonic solid is a polyhedron which a. has only one kind of regular polygon for a face and b. has the same number of faces at each vertex.

One simple example is that of a cube. Each face is a square (=regular quadrilateral) and each vertex is connected to exactly three squares.

In Schläfli notation, this is {4, 3}: the regular polyhedron with 3 regular 4-gons at each vertex.

- Can there be polyhedra with exactly one or two squares at each vertex?
- Can there be polyhedra with exactly four squares at each vertex?

For future reference, we'll need the notion of the "angular defect" as well. Rather than focusing on what is *present* at the vertex, we focus on what is *absent*: the deviation from 360 degrees at the vertex, or "leftover" angle. In this case, it is $360 - 90 \times 3 = 90$.

Let’s prove that there are exactly 5 Platonic solids and no more.

- Be careful – we’ll need some way to handle discussing regular n-gons as n becomes large.

Solid	**Schläfli #**	**Defect at vertex**	**V**
Tetrahedron			
Cube			
Octahedron			
Dodecahedron			
Icosahedron			

Q: Can you find the combinatorial properties of a regular polytope from the Schläfli number alone?

Part 3 – Duals, Truncations, Stellations, χ_E

Let’s try placing a point in the middle of each face and connecting the points. Maybe something interesting will happen.

Write a sentence explaining the pattern that you found:

Q: What happens to the Schläfli number of a polyhedron when you take the dual?

Two common operations on polyhedra, in addition to taking the dual, are stellation (=replacing each face with a pyramid) and truncation (=replacing each vertex with an appropriate regular polygon). Let’s experiment with the snub and stellated cubes:

- Same as before: let’s find the vertices, edges, faces, and angular defect at each vertex.

Try something more complicated: the truncated icosahedron, also known as a “soccer ball.”

- Control question – which polygon replaces each vertex?

We’ve produced some data. Let’s put it in a convenient table. Do you know the Euler Characteristic? While we’re at it, we might as well throw that in too.

Solid	**Defect at vertex**	**V**	**Total Defect at all V**	$X_E = V - E + F$
Tetrahedron				
Cube				
Octahedron				
Dodecahedron				
Icosahedron				
Snub Cube				
Stellated Cube				
Soccer Ball				

Do you see a pattern? Describe it:

Part 4 – Euler Characteristic and Gauss-Bonnet Theorem

We may have a nice pattern over the figures that we've put in the table above, but it has only 8 entries! It would be great to have something a little bit more convincing.

Let's try to find the total angular defect and Euler Characteristic of, say, a donut shape built from cubes.

- Is this the same as finding the same values for a torus? Why or why not?

Total Angular Defect:

Euler Characteristic:

Our hypothesis:

Proof:

- Hint: Parts 1 and 2 will be useful here.

Q (topology): Can you find the Euler Characteristic of a Möbius strip? A Klein bottle?

The angular defect is not merely a nice tool for finding the certain features of polyhedra: it describes curvature and helps to explain how map distortions are created.

Topologically speaking, we will find that all polyhedra with equal euler characteristic are equivalent.

Further reading:

- Alexander Givental, “Geometry of Surfaces and the Gauss-Bonnet Theorem:” https://math.berkeley.edu/~giventh/difgem.pdf
- Coxeter, *Regular Polytopes*.
- Ueno, Shiga, Morita, *A Mathematical Gift, I: The interplay between topology, functions, geometry, and algebra*