PROBLEM STRUCTURES

IGOR GANICHEV

BMC, April 17th, 2019

1. Lines

Consider a 3×3 square, whose each cell can contain either $+$ or $-$. Call a particular selection of signs for all cells a "configuration". We are allowed to perform the following operation. Choose a row or a column and invert all signs in it. Using these operations, we can transform a given configuration into many other configurations. Our goal is to understand the structure of these configurations.

Problem 1.1. Given a configuration, is it always possible to transform it to any other configuration?

Useful questions to consider:

- Consider answering the same questions for 2×2 square.
- Is the order of operations important?
- Do we need to consider transformations where the same operation is performed multiple times?
- What is the number of possible configurations?
- What is the number of transformations that can potentially result in a unique configuration?
- If we can transform configuration A into configuration B, can we always transform B into A?

Problem 1.2. Let M be the configuration with a $-$ in all cells. a) How many configurations can be reached from M? b) Assuming that no row/column is inverted more than once, how many different transformations (ignoring order) can produce the same configuration starting from M?

Problem 1.3. Let M be the configuration with a $-$ in all cells. Which numbers of pluses can appear in a configuration reachable from M?

Problem 1.4. How can we describe the space of all possible configurations when viewed from the perspective of reachability? Can we easily categorize the space of configurations for 2×2 squares?

Problem 1.5. Can we come up with a simple test or a set of tests to decide whether two given configurations are connected (one can be reached from another)?

2. Crosses

Consider a 4×4 square, whose each cell can contain either $+$ or $-$. Call a particular selection of signs for all cells a "configuration". We are allowed to perform the following operation. Select a cell and invert all signs in its row and column.

Problem 2.1. Can we transform a square full of pluses into a square full of minuses?

Problem 2.2. Can we transform a square with one row of pluses (the rest being minuses) into a square full of minuses?

Problem 2.3. Can we transform a square with a plus in the bottom left corner into a square full of minuses?

Problem 2.4. Can we transform any configuration into any other configuration?

3. Neighbors

Consider a 7×7 square, whose each cell can contain either $+$ or $-$. Call a particular selection of signs for all cells a "configuration". We are allowed to perform the following operation. Select a cell and invert the signs in it and its neighbors. Two cells are neighbors if they share a point.

Problem 3.1. Can any configuration can be transformed into any other configuration?