FUNCTIONAL EQUATIONS

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1. Introduction

Definition 1.1. Let $f : A \to B$ be a function. The set A is called the *domain*, and B the *codomain*.

Definition 1.2. A function $f : A \to B$ is *injective* if $f(x) = f(y) \iff x = y$. (Sometimes also called *one-to-one*.)

Definition 1.3. A function : $A \to B$ is *surjective* if for all $b \in B$, there is some $x \in A$ such that $f(x) = b$. (Sometimes also called *onto*.)

Definition 1.4. A function is *bijective* if it is both injective and surjective.

2. Problems

Problem 2.1. Find all functions $f : \mathbb{R} \to \mathbb{R}$ such that for all $x \in \mathbb{R}$

$$f(x) + 2f(-x) = 3x^2 + x.$$

Problem 2.2. Find all functions $f : \mathbb{R} \to \mathbb{R}$ such that for all $x \in \mathbb{R}$

$$f(2x + 1) = x^2.$$

Problem 2.3. Find all functions $f : \mathbb{R} \to \mathbb{R}$ such that for all $x \in \mathbb{R}$

$$f(x)^2 = x^2.$$

Problem 2.4. Find all functions $f : \mathbb{R} \to \mathbb{R}$ such that for all $x, y \in \mathbb{R}$

$$f(x + y) = f(x) + y.$$

Problem 2.5. Find all functions $f : \mathbb{R} \to \mathbb{R}$ such that for all $x, y \in \mathbb{R}$

$$f(x + y) + f(x - y) = 2x^2 + 2y^2.$$

Problem 2.6. Find all functions $f : \mathbb{R} \to \mathbb{R}$ such that for all $x \in \mathbb{R}$, $x \neq 0$

$$f(x) + 2f(\frac{1}{x}) = 3x.$$

Problem 2.7. If function $f : \mathbb{R} \to \mathbb{R}$ is strictly monotonic, what can we say about $f(f(x))$?

Problem 2.8. If function $f : \mathbb{R} \to \mathbb{R}$ satisfies $f(f(x)) = x$ for all $x \in \mathbb{R}$, show that it is bijective.

Problem 2.9. Find all functions $f : \mathbb{R} \to \mathbb{R}$ such that for all $x \in \mathbb{R}$

$$f(yf(x) + xy) = 2xy$$

$$f(f(x)) = x$$

Problem 2.10. Find all functions $f : \mathbb{R} \to \mathbb{R}$ such that for all $x \in \mathbb{R}$

$$f(f(x)^2 + f(y)) = xf(x) + y$$

Problem 2.11. Find all functions $f : \mathbb{R} \to \mathbb{R}$ such that for all $x \in \mathbb{R}$

$$f(f(y)x^2) = xf(y) + 1$$

Problem 2.12 (Cauchy's Functional Equation over $\mathbb{Q}$)**.** Find all functions $f : \mathbb{Q} \to \mathbb{Q}$ such that for all $x \in \mathbb{Q}$

$$f(x + y) = f(x) + f(y)$$

3. Extra Problems

Problem 3.1. Find all functions $f : \mathbb{R} \to \mathbb{R}$ such that for all $x \in \mathbb{R}$, $x \neq 0$

$$f(x) + f(1 - \frac{1}{x}) = 2x$$

Problem 3.2. Find all functions $f : \mathbb{R} \to \mathbb{R}$ such that for all $x \in \mathbb{R}$

$$(x - y)f(x + y) - (x + y)f(x - y) = 4xy(x^2 - y^2)$$