BMC Spring 2019
Beginners

Mathe-Magical Scissors: Equidecomposition of Polygons Part III

By Harry Main-Luu

Recall that we were cutting things into squares!
We will finish that idea today, generalize that practice to cut any polygons, and then discuss some things that are impossible.

I. More on Rectangles - Truth or Lies

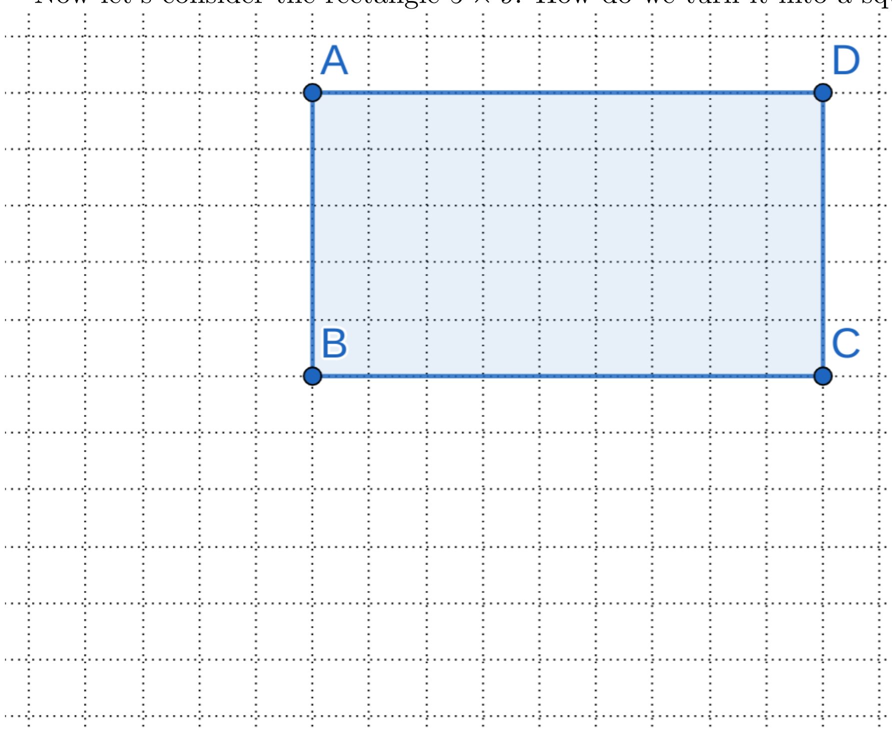
We all remember how to cut the simple 5×13 rectangle.

Let's carefully examine the method of decomping the rectangle to a square. Does it ALWAYS work?

1. Draw a rectangle 1×9. Perform the construction for square root as we learned.
2. If you are lucky (and precise), your square root should be a familiar number (but we already know that). Now does the 1×9 rectangle decompose into the 4 pieces as we would hope?

As it turns out, the decomposition that we described last time only works for certain rectangles, but not all. We want a more fool-proof method.

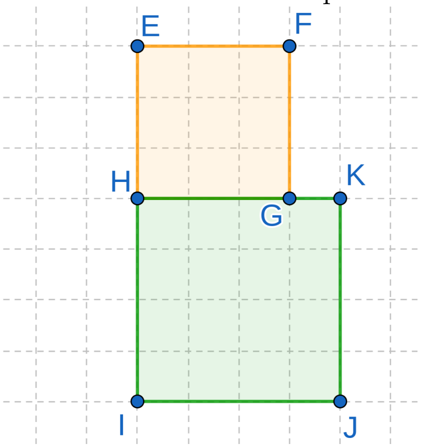
Now let's consider the rectangle 5×9. How do we turn it into a square?



Could there be a glitch in this approach as well? Try the 2×9. How do you propose we fix this glitch?

II. Combining Squares

Now that we are able to cut any shapes and put them into squares. Let's try to put some squares together. We briefly discuss this last time.
Are we able to cut two squares and put them together into one?



1. How are we so sure that the construction given works exactly the way we want it to? Where do you see a potential problem?

III. Summary - Main Results

Remember the original question?

> Given a hexagon and a pentagon of the same area. Is it possible to cut the hexagon into finitely many pieces and glue it back into the pentagon perfectly, where no pieces overlap and we use all of the pieces?

Let's summarize what we have accomplished:

1. We can decompose any triangle into a rectangle.

2. We can decompose any rectangle into a square.

3. We can combine any two squares into one bigger square.

4. Are these facts enough? What else are we missing?

Here's the Big Theorem:

> **Theorem III.1** (Bolyai (1832) - Gerwien (1833)). Two polygons are equidecomposable if and only if they have the same area.

Proof. For each of the polygons, say P_1, P_2, apply the following procedure:

(a) Triangulate the polygon.

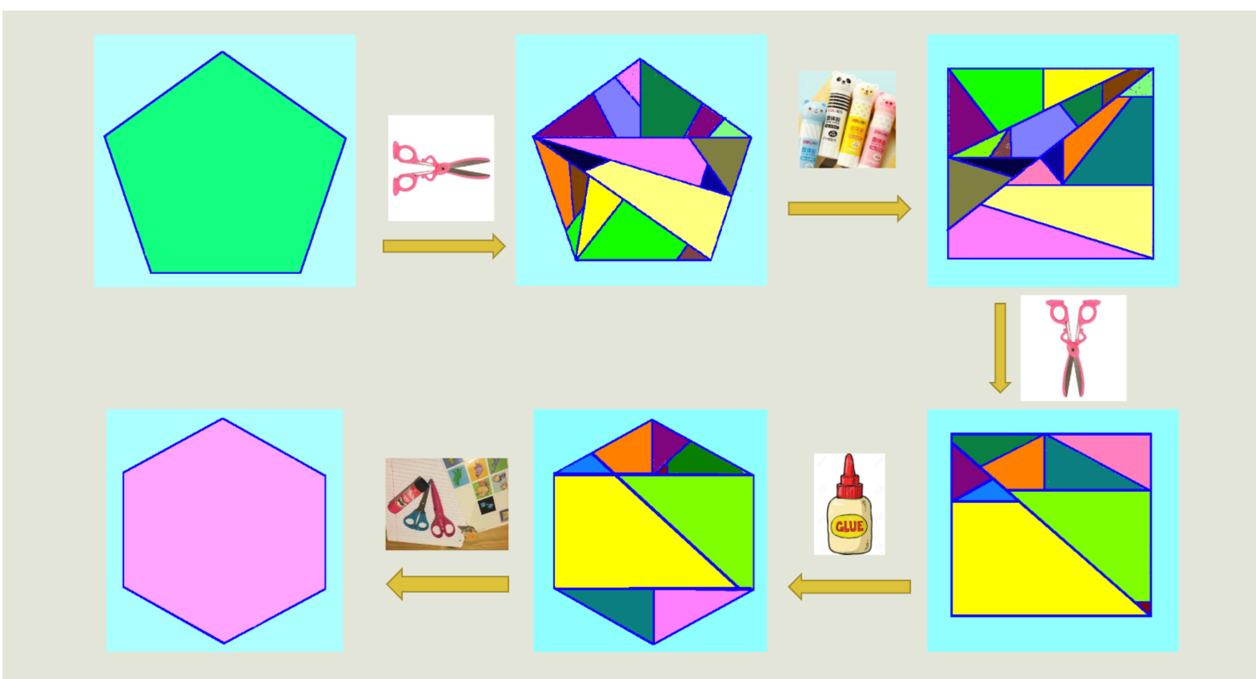
(b) Decompose all of triangles into rectangles, and then into squares.

(c) Combine all of the squares (since there are finitely many of them) into a big square of the same area as the polygon.

(d) Call S_1 the resulting square from decomposing P_1, and S_2 similarly.

Note that the dimension of the two squares must be the same since they have the same area. Hence the two squares are congruent. Combining all of the cuts made in the two procedures yields the desired result. □

Note that the decomposition is not unique. The picture below describes one way of decomposing the 5-gon and the 6-gon, if they have the same area.



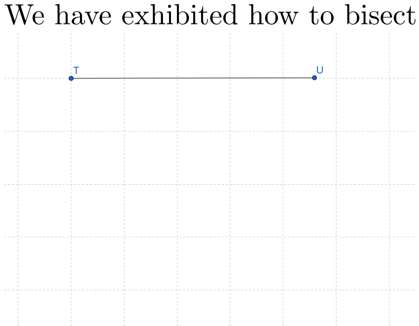
IV. Applications

What can we do with all of this new knowledge that we have learned?
For example, we can make puzzles and games for children (and adults), such as a puzzle kit of, say 20 pieces, where they can fit in any of the premade molds of a square, a rectangle, a hexagon, or a pentagon (as long as we make the molds of the same area).
Any other type of games you can think of?

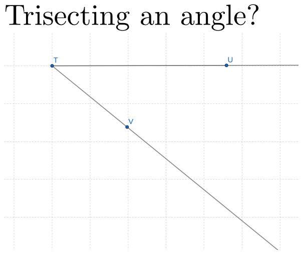
V. Further discussion

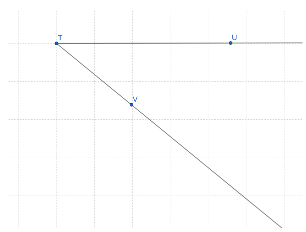
We have discussed the possibilities of many decompositions and constructions. In mathematics, it is generally easier to show that a task is possible by just exhibiting one way of accomplishing it. However, proving something is impossible is a completely different animal. How would you know that something is impossible just because you have yet to find the solution. It could very well be that the next day someone else across the globe finds the solution you've been looking for.
We will now explore some of the impossibility (limitations) of the compass and straightedge. As mentioned, some of these took mathematicians thousands of years to prove their impossibility.

1. We have exhibited how to bisect a segment. Can we trisect a segment?



2. How about bisecting an angle?
   Trisecting an angle?





3. The answer to the previous problem relies on being able to construct the cubic root of numbers (why?). Let's try to construct the simplest (non-trivial) cubic root: $\sqrt[3]{2}$.
   This problem has a cute name: Doubling the cube.

There are many problems of this nature, such as squaring a circle (drawing a circle and a square of same area). We can also try to generalize the question of decomposition to higher dimensions, ie Can we decompose two polyhedra of the same volume into finitely many pieces? Or, we can also restrict our decomposition and see if it still works, such a question can be "Can we decompose two polygons without completely cutting them apart (every piece is still connected to another piece at one corner, forming a chain of smaller pieces)?" This is called "hinged decomposition".
The world of mathematics is so vast, even such a small topic still has many possibilities. I hope everyone gets to explore and continue to explore many of these fun maths. Until next time!