Definite Integration with Complex Analysis

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Definition 1. $e = \lim_{n\to\infty} \left(1 + \frac{1}{n}\right)^n$

Definition 2. $e^{ix} = \cos x + i \sin x$ for $x \in \mathbb{R}$, and $e^{a+bi} = e^a \cdot e^{bi}$ for $a, b \in \mathbb{R}$.

Note that $\left|e^{a+bi}\right| = e^a$ and that $\arg\left(e^{a+bi}\right) \equiv b$.

Exercise 1. Verify $e^{z+w} = e^z \cdot e^w$ for $z, w \in \mathbb{C}$.

Definition 3. $\cos(z) = \dfrac{e^{iz} + e^{-iz}}{2},\ \sin(z) = \dfrac{e^{iz} - e^{-iz}}{2i}\ \ \forall\, z \in \mathbb{C}$.

Differentiation

Definition 4. If it exists, the *derivative* of a function f at z is defined to be

$$f'(z) = \lim_{\epsilon\to 0} \frac{f(z+\epsilon) - f(z)}{\epsilon},$$

and f is said to be *differentiable* at z.

As ϵ can approach 0 can approach from any direction in the complex plane, differentiability is a stronger condition for complex numbers than for real numbers.
As in calculus,

1. $f \mapsto f'$ is a linear transformation, i.e., $(Cf)' = Cf'$ and $(f+g)' = f' + g'$
2. $(fg)' = f'g + fg'$ (product rule)
3. $(f/g)' = \frac{f'g - fg'}{g^2}$ (quotient rule)
4. $g \circ f = (g' \circ f)(f')$

Exercise 2. Show that $(z^n)' = nz^{n-1}$ for $z \in \mathbb{C},\ n \in \mathbb{N}$. What about $n \in \mathbb{Z}$?

Exercise 3. Show that $(e^z)' = e^z$ for $z \in \mathbb{C}$.

Definition 5. A function f is *holomorphic* on an open set $U \in \mathbb{C}$ if $f(z)$ is differentiable for all $z \in U$.

Conformality

Definition 6. A *path* in an open set U is a function $[a, b] \to U$ that is differentiable except at a finite number of points.

Definition 7. Let f be holomorphic on the open set $U \in \mathbb{C}$, and let $\gamma : [a, b] \to U, \eta : [c, d] \to U$ for $a, b, c, d \in \mathbb{R}$ be paths in U. The *tangent* at $x \in [a, b]$ to γ is defined to be $\gamma'(x)$ if it exists and is nonzero.

Definition 8. The *angle* from γ to η at (t_0, t_1) is defined to the angle from the tangent vector of γ at t_0 to η at t_1, i.e., $\arg\left(\dfrac{\eta'(t_1)}{\gamma'(t_0)}\right)$, if $\gamma(t_0) = \eta(t_1)$ and $\gamma'(t_0), \eta'(t_1) \neq 0$.

Exercise 4. Prove that if the angle from γ to η is defined at (t_0, t_1) and $f'(\gamma(t_0)) = f'(\eta(t_1)) \neq 0$, it is equal to the angle from $f \circ \gamma$ to $f \circ \eta$ at (t_0, t_1).

Integration

Definition 9. The integral of $f : U \to \mathbb{C}$ over a differentiable (except at a finite number of points) path $\gamma : [a, b] \to U$ for $a, b \in \mathbb{R}$ is

$$\int_\gamma f \, \mathrm{d}z = \int_a^b (f \circ \gamma)(\gamma') \, \mathrm{d}t.$$

This is one of many ways to define a complex integral. I use it so that things we know from calculus have similar proofs:

Exercise 5. Show that $f \mapsto \int_\gamma f$ is a linear transformation, i.e.,

$$\int_\gamma Cf(z) \, \mathrm{d}z = C \int_\gamma f(z) \, \mathrm{d}z \qquad \text{and} \qquad \int_\gamma f(z) + g(z) \, \mathrm{d}z = \int_\gamma f(z) \, \mathrm{d}z + \int_\gamma g(z) \, \mathrm{d}z.$$

Exercise 6. Given $\gamma : [a, b] \to U$ and $\eta : [b, c] \to U$ such that $\gamma(b) = \eta(b)$, denote by $\gamma \wedge \eta : [a, c] \to U$ the path $\gamma \wedge \eta = \begin{cases} \gamma & \text{if } x \in [a, b] \\ \eta & \text{if } x \in [b, c] \end{cases}$. Show that $\displaystyle\int_\gamma f \, \mathrm{d}z + \int_\eta f \, \mathrm{d}z = \int_{\gamma \wedge \eta} f \, \mathrm{d}z$.

Exercise 7. Write $\displaystyle -\int_\gamma f(z) \, \mathrm{d}z$ as $\displaystyle\int_\eta f(z) \, \mathrm{d}z$ for some η.

Exercise 8. Prove that a reparameterization of γ doesn't change $\int_\gamma f \, \mathrm{d}z$, i.e. that if differentiable (except at a finite number of points) $\gamma : [a, b] \to U, g : [c, d] \to [a, b]$ satisfy $f(c) = a, f(d) = b$, then

$$\int_\gamma f \, \mathrm{d}z = \int_{\gamma \circ g} f \, \mathrm{d}z.$$

Exercise 9. Let F be holomorphic on U, and let $\gamma : [a, b] \to U$ for $a, b \in \mathbb{R}$ be a path in U. Show that

$$\int_\gamma F' \, \mathrm{d}z = F(\gamma(b)) - F(\gamma(a)).$$

Definition 10. A path $\gamma : [a, b] \to U$ is *closed* if its *endpoints*, $\gamma(a)$ and $\gamma(b)$, are equal.

Exercise 10. Let C_R be the closed path formed by the counterclockwise circle of radius R centered at the origin. Evaluate $\displaystyle\int_{C_R} z^n \, \mathrm{d}z$ for $n \in \mathbb{Z}$.

Path Indepedence

Definition 11. A function $f : U \to \mathbb{C}$ is *path-independent* if $\int_\gamma f \, \mathrm{d}z = \int_\eta f \, \mathrm{d}z$ whenever the corresponding endpoints of γ and η are the same.

Exercise 11. Show that the following three conditions are equivalent for U an open set and $f : U \mapsto \mathbb{C}$:

1. f is path-independent,
2. $\int_\gamma = 0$ for any closed path $\gamma \in U$, and
3. there exists a function F such that $F' = f$.

Definition 12. The *length* of a path $\gamma : [a, b] \to U$, denoted $L(\gamma)$, is $\displaystyle\int_\gamma |\mathrm{d}z| = \int_a^b \left|\gamma'(t)\right| \mathrm{d}t$.

Exercise 12. Show that $\displaystyle\left|\int_\gamma f\right| \leqq L(\gamma) \max_{t \in [a,b]} |f(\gamma(t))|$.

Definition 13. A sequence of functions $f_n \mid n \in \mathbb{N}$ converges uniformly if for all $\epsilon > 0$ there exists $N \in \mathbb{N}$ such that for $n > N$, $|f_n - f| < \epsilon$.

Exercise 13. Show that if $f_n \mid n \in \mathbb{N}$ converges uniformly on an open set U, $\lim_{n\to\infty} \int_\gamma f_n = \int_\gamma f$ for any path γ.

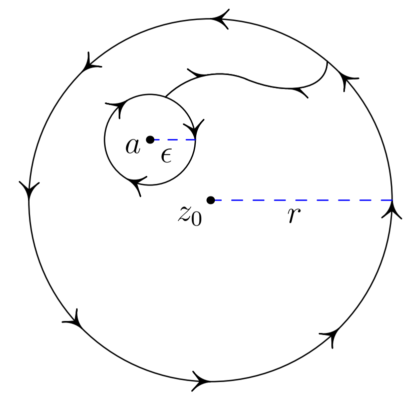
Definition 14. An open set U is *homeomorphic to a disc* if there is a continuous bijection $f : D \mapsto U$ with a continuous inverse, where D is the open unit disc.

Theorem 1. *Let R be the counterclockwise boundary of a axis-parallel rectangle in the complex plane in an open set U homeomorphic to a disc. Let f be holomorphic on U. Then,*

$$\int_R f \,\mathrm{d}z = 0.$$

Exercise 14. Show that if f is holomorphic on U, f has an anti-derivative F such that $F' = f$, and thus, that f is path independent.

Cauchy's Integration Formula



Exercise 15. Let $C_{R,z}$ be the path formed by the counterclockwise circle of radius R centered at z. Prove Cauchy's Integration Formula,

$$\int_{C_{r,z_0}} \frac{f(z)}{z-a}\,\mathrm{d}z = 2\pi i\, f(a),$$

for $|a - z_0| < r < R$ and f holomorphic on an open set containing $|z - z_0| < R$.

[*Hint*: Use the path on the right.]

Exercise 16. Evaluate $\displaystyle\int_{C_1} \frac{0.3\cos\left(16.43^{z^2\sin(\pi^2 z)}\right)}{(6z+4z^2-z^3)e^{8z/\pi}}\,\mathrm{d}z.$

Exercise 17. Let $f_n \,\Big|\, n \in \mathbb{N}$ be a sequence of holomorphic functions converging uniformly on every compact (closed and bounded) subset of an open set U. Show that f_n' also converges uniformly to f' on every compact subset..

Power Series

Definition 15. A *Taylor series* is an expression of the form $\displaystyle\sum_{n=0}^{\infty} a_n(z-z_0)^n$ that converges for $|z - z_0| < R$ for some $R > 0$. R is then said to be *a radius of convergence*, and the maximum such R (possibly ∞) is *<u>the</u> radius of convergence*.

Exercise 18. Show that if a Taylor series equals zero, it is 0.

Exercise 19. Let f be holomorphic on an open set containing $|z - z_0| \le R$. Show that f is equal to a unique Taylor series with R a radius of convergence for $|z - z_0| < R$.

$\left[\textit{Hint 1: } f(z) = \dfrac{1}{2\pi i}\displaystyle\int_{C_{r,z}} \frac{f(\zeta)}{\zeta - z}\,\mathrm{d}\zeta.\right]$ $\left[\textit{Hint 2: }\text{Show that } \dfrac{1}{\zeta - z} = \dfrac{1}{\zeta - z_0}\left(\displaystyle\sum_{n=0}^{\infty}\left(\frac{z-z_0}{\zeta-z_0}\right)^n\right).\right]$

Exercise 20. Show that $\dfrac{1}{n!} f^{(n)}(z_0) = \dfrac{1}{2\pi i}\displaystyle\int_{C_{R,z}} \frac{f(\zeta)}{(\zeta - z)^{n+1}}\,\mathrm{d}\zeta$. (Here, $f^{(n)}$ means the n^{th} derivative of f.)

Exercise 21. Evaluate $\displaystyle\int_{C_1} \frac{\cos e^{-z}}{z^2}\,\mathrm{d}z.$

Exercise 22. Evaluate $\displaystyle\int_{C_2} \left(\frac{z}{z-1}\right)^{2019}$.

Exercise 23. Show that if f is holomorphic and bounded on $\mathbb{C}$, it is constant.

Exercise 24. Prove the fundamental theorem of algebra (stating that all non-trivial polynomials have roots).

Definition 16. A *Laurent series* is an expression of the form $\displaystyle\sum_{n\in\mathbb{Z}} a_n(z-z_0)^n$ that converges for $r < |z-z_0| < R$ for some $R > r \geq 0$.

Exercise 25. Show that if a Laurent series equals zero, it is 0.

Exercise 26. Show that if f is holomorphic on an open set containing $r \leq |z-z_0| \leq R$, then

$$f(z) = \frac{1}{2\pi i}\int_{C_{R,z_0}} \frac{f(\zeta)}{\zeta - z}\,\mathrm{d}\zeta - \frac{1}{2\pi i}\int_{C_{r,z_0}} \frac{f(\zeta)}{\zeta - z}\,\mathrm{d}\zeta$$

$\forall z \;\Big|\; r < |z-z_0| < R$.

Exercise 27. Let f be holomorphic on an open set containing $r \leq |z-z_0| \leq R$. Show that f is equal to a unique Laurent series for $r < |z-z_0| < R$.

Residues

Definition 17. Given the Laurent series $f(z) = \displaystyle\sum_{n\in\mathbb{Z}} a_n(z-z_0)^n$, the *residue* of f at z_0 is $\operatorname{Res}_{z_0} f = a_{-1}$.

Exercise 28. Show that for a Laurent series $f(z) = \displaystyle\sum_{n\in\mathbb{Z}} a_n(z-z_0)^n$ converging for $0 < |z-z_0| < R$,

$$\int_{C_{R,z_0}} f(z)\,\mathrm{d}z = 2\pi i \operatorname{Res}_{z_0} f.$$

Definition 18. The *winding number* of a closed path γ around z, where z is not in the image of γ, is

$$W(\gamma, z) = \frac{1}{2\pi i}\int_\gamma \frac{1}{z}\,\mathrm{d}z.$$

(This is the number of times that γ wraps around z_0.)

Exercise 29. Let U be homeomorphic to an open disc and let γ be a closed path in U. Show that if f is holomorphic on $U \setminus \{z_1, \ldots, z_n\}$,

$$\int_\gamma f\,\mathrm{d}z = 2\pi i \sum_{j=1}^{n} W(\gamma, z_j)\operatorname{Res}_{z_j} f.$$

Definition 19. A Laurent series f has a *simple pole* at z_0 if it is of the form $f(z) = \displaystyle\sum_{n=-1}^{\infty} a_n(z-z_0)^n$.

Exercise 30. Let f have a simple pole at z_0 and let g be holomorphic on an open set containing z_0. Show that

$$\operatorname{Res}_{z_0}(fg) = g(z_0)\operatorname{Res}_{z_0} f.$$

Exercise 31. Show that if f is holomorphic on an open set containing z_0, $f(z_0) = 0$, and $f'(z_0) \neq 0$, then $\dfrac{1}{f}$ has a simple pole at z_0 of residue $\operatorname{Res}_{z_0}\dfrac{1}{f} = \dfrac{1}{f'(z_0)}$.

Exercise 32. Evaluate $\displaystyle\int_{C_1} \cot(z)\,\mathrm{d}z$.

Exercise 33. Evaluate $\displaystyle\int_{C_{10}} \frac{1-z}{1-e^z}\,\mathrm{d}z$.

Definite Integration

Trigonometric

Exercise 34. Let D be the open unit disc, $\left\{z \,\middle|\, |z| < 1\right\}$, and let $\overline{D}$ be the closed unit disc. Let

$$f(z) = \frac{Q\left(\frac{1}{2}\left(z + \frac{1}{z}\right), \frac{1}{2i}\left(z - \frac{1}{z}\right)\right)}{iz}.$$

Show that if f is holomorphic on an open set containing $\overline{D} \setminus S$, where S is a finite subset of D,

$$\int_0^{2\pi} Q(\cos\theta, \sin\theta)\, \mathrm{d}\theta = 2\pi i \sum_{z \in S} \mathrm{Res}_z\, f.$$

Exercise 35. Evaluate $\displaystyle\int_0^{2\pi} \frac{1}{a + b\cos\theta}\, \mathrm{d}\theta$, where $0 < |b| < a$.

Exercise 36. Evaluate $\displaystyle\int_0^{2\pi} (\cos\theta)^n\, \mathrm{d}\theta$ for $n \in \mathbb{N}$.

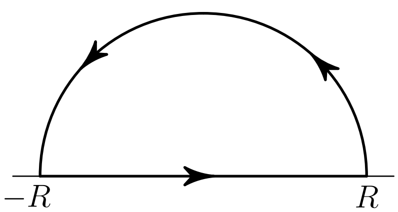
Exercise 37. Evaluate $\displaystyle\int_0^{2\pi} \frac{1}{1 + a^2 - 2a\cos\theta}\, \mathrm{d}\theta$ for $0 < a < 1$.

Improper Integrals

Exercise 38. Let H be the open upper half plane, $\{a + bi \mid a, b \in \mathbb{R}, b > 0\}$, and let $\overline{H}$ be the closed upper half plane. Show that if f is holomorphic on an open set containing $\overline{H} \setminus S$, where S is a finite subset of H, and $\lim_{z \to \infty} z f(z) = 0$, then

$$\int_{-\infty}^{\infty} f(x)\, \mathrm{d}x = 2\pi i \sum_{z \in S} \mathrm{Res}_z\, f.$$

[*Hint*: Use the path on the right.]



Exercise 39. Find $\displaystyle\int_{-\infty}^{\infty} \frac{x^2}{1 + x^6}\, \mathrm{d}x$.

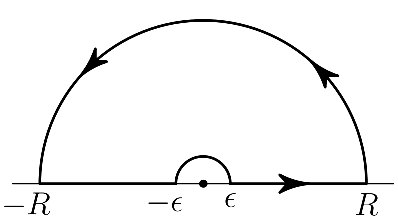
Exercise 40. Find $\displaystyle\int_0^{\infty} \frac{1}{1 + x^n}\, \mathrm{d}x$ for $2 \le n \in \mathbb{N}$.

[*Hint*: Use the closed path from 0 to R to $Re^{2\pi i/n}$ and back.]

Note: You would not want to evaluate this indefinitely. For $n = 50$, Wolfram Alpha gives an integral 3000 characters long, and Wolfram Alpha Pro Premium cannot find the integral for $n = 100$.

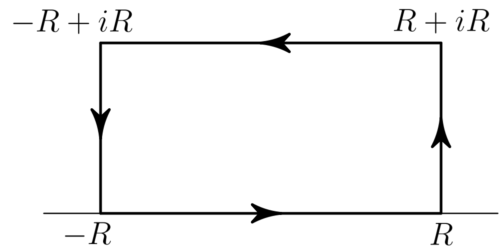
Exercise 41. Evaluate $\displaystyle\int_0^{\infty} \frac{(\log x)^2}{1 + x^2}\, \mathrm{d}x$.

[*Hint*: Use the contour on the right.]



Fourier Transforms

Exercise 42. Let H be the open upper half plane, $\{a + bi \mid a, b \in \mathbb{R}, b > 0\}$, and let $\overline{H}$ be the closed upper half plane. Show that if f is holomorphic on an open set containing $\overline{H} \setminus S$, where S is a finite subset of H, and $zf(z)$ is bounded for large z, then for $0 < a \in \mathbb{R}$,



$$\int_{-\infty}^{\infty} f(x)e^{iax}\,\mathrm{d}x = 2\pi i \sum_{z\in S} \mathrm{Res}_z\, e^{iax} f.$$

[*Hint*: Use the path on the right.]

Exercise 43. Evaluate $\displaystyle\int_{-\infty}^{\infty} \frac{\cos x}{a + x^2}\,\mathrm{d}x$ for $0 < a \in \mathbb{R}$.

Mellin Transforms

Definition 20. For $z \in \mathbb{C}$, let $t \;\Big|\; 0 \le \operatorname{Im} t < 2\pi$ be such that $e^t = z$. When then write $z^a = e^{at}$.

Exercise 44. Let f be holomorphic on an open set containing $\mathbb{C} \setminus S$, where S is a finite subset of $\mathbb{C}$. Show that if $\displaystyle\lim_{z\to(0\vee\infty)} z^a f(z) = 0$, then

$$\int_0^{\infty} f(x)x^a \frac{\mathrm{d}x}{x} = -\frac{\pi e^{-\pi i a}}{\sin \pi a} \sum_{z\in S} \mathrm{Res}_z\, z^{a-1} f.$$

Exercise 45. Given the same condition, show that $\displaystyle\int_0^{\infty} f(x)x^a \frac{\mathrm{d}x}{x} = \int_{-\infty}^{\infty} f(e^t)e^{at}\,\mathrm{d}t$.

Exercise 46. Evaluate $\displaystyle\int_0^{\infty} \frac{x^a}{1 + x^n}\frac{\mathrm{d}x}{x}$ for $0 < a < 1,\ n = 1$. For $n \in \mathbb{N}$?