Part A: Some Counting Techniques

Because counting comes up in many different contexts, it is useful to use the language of sets to describe the principles and tricks of the trade, which can then be applied in all these contexts.

Notation

The notation $S = \{1, 4, 9\}$ means that *S* is the set consisting of the three elements 1, 4 and 9. We say that 1 is an element of *S*, or 1 belongs to *S*, and we write $1 \in S$. Since 2 is not an element of *S*, we write $2 \notin S$. We also write $|S| = 3$ to mean that *S* has exactly three elements.

If $T = \{4, 9\}$ we say that *T* is a subset of *S*, because every element of *T* is also in *S*. We write $T \subset S$ but some people write $T \subseteq S$.

There is a tricky set, called the "empty set", denoted by ϕ. It is the set with no elements! It is a subset of every set.

1. How many subsets does the set $S = \{1, 4, 9\}$ have? List them all using set notation. How many of the subsets have no elements? Exactly one element? Exactly two elements? Exactly three elements?

The Counting Principle

2. Suppose you have 3 shirts (one yellow, one green and one black) and two pants (one blue and one brown) and you need to choose one shirt and one pants to wear.
In how many different ways can you get dressed?

• Make a systematic list of all the choices. Is there another "logical" way of listing them?
• Arrange the choices in a rectangular array that "makes sense".
• Make a "branching tree" diagram showing all the choices. Now make a different tree diagram showing all the choices.

3. Now suppose that, in addition to the two choices you made in problem 2, you also have to choose one hat among two hats that you have, one grey and one red.
Now in how many ways can you get dressed (shirt, pants and hat)?

- Make a systematic list.
- Make a tree diagram of all the choices.
- Discuss how you can get the answer to problem 3 using your answer to problem 2.

4. Now generalize the above two problems to state the <u>counting principle</u>:
Suppose you have to carry out an activity consisting of *k* parts or steps or stages. The first stage can be done in a_1 ways, the second stage can be done in a_2 ways, and so on until the *k*th stage, which can be done in a_k ways. In how many ways can the whole activity be done?

Explain why the counting principle is also called the "multiplication" principle.

5. Suppose you roll a pair of normal dice. How many possible outcomes are there? Here an outcome is an *ordered* pair of numbers because we distinguish between the first and second die. For example, (3,4) and (4,3) are two *different* possible outcomes: in (3,4) the first die turned up 3 and the second die turned up 4, while in (4,3) the first die turned up 4 and the second die turned up 3. If it helps, you can think of one die being blue and the other red.
- Make a square array showing all possible outcomes.
- How many outcomes give you a sum of 1? 2? 3? 4? 5? 6? 7? 8? 9? 10? 11? 12? 13? Can you find them easily in your square array?

6. Suppose you flip a coin ten times. How many possible outcomes are there? Here an outcome is a string of Hs and Ts. For example, one possible outcome is HHTHTTTHHH.

7. Use the counting principle to explain why if a set has (exactly) *n* elements, then it has 2^n subsets. Recall that $2^n = 2 \times 2 \times \cdots \times 2$ (*n* factors). Does this agree with your answer to problem 1 above?
Hint: each subset corresponds to making *n* choices: to be, or not to be? That is the question.

8. As you know, every integer $n > 1$ can be written uniquely (except for the order of the factors) as a product of primes. Suppose that $n = p_1^{a_1} p_2^{a_2} \cdots p_k^{a_k}$ is the prime factorization of *n*, where $p_1, p_2, \ldots, p_k$ are its distinct prime factors. Can you find how many divisors (factors) *n* has? [The divisors of 6, to give an example, are 1, 2, 3 and 6. So 6 has four divisors or factors].

9. Looking at your answer to question 8, can you figure out which positive integers have an odd number of factors?

10. In a school there are 100 lockers, numbered from 1 to 100. There are also 100 students, numbered from 1 to 100. The lockers are all initially closed. The first student (student number 1) opens all the lockers. That is, she changes all the lockers that are multiples of 1. Then the second student (student number 2) closes every other locker. That is, he changes all the lockers that are multiples of 2. And so on. That is, for every *n* from 1 to 100, student number *n* changes all the lockers that are multiples of *n*. To change a locker means to open it if it is closed, and to close it if it is open.

After all the students go through the lockers, what lockers will be open?

The Addition Principle

We need a little more notation. The union of two or more sets is the set consisting of all elements that belong to at least one of the sets. We write $A \cup B, A \cup B \cup C,$ etc.
The intersection of two or more sets is the set consisting of all elements that belong to all of the sets. We write $A \cap B, A \cap B \cap C,$ etc. Two sets are disjoint if they have no elements in common. That is, if their intersection is the empty set.

The addition principle says that if we have a collection of pairwise disjoint sets (that is, no matter which two of them you pick, they have no elements in common), then the number of elements in their union is simply the sum of the number of elements in each set:
$|S_1 \cup S_2 \cup \cdots \cup S_k| = |S_1| + |S_2| + \cdots + |S_k|$. (Here the sets are pairwise disjoint).

11. Suppose a restaurant offers 5 appetizers, 6 main entrees and 3 desserts. How many different two--course meals can be chosen, where a two course meal is understood to include a main entrée? What if a two-course meal does not have to include a main entrée?

The Inclusion-Exclusion Principle

Remember that the addition principle only works when the sets are pairwise disjoint.

12. Explain why if two (finite) sets *A* and *B* have some elements in common, then it is *no longer true* that $|A \cup B| = |A| + |B|$. Give an example and draw a Venn diagram of your example. Can you "fix" the problem? That is, can you find a formula for $|A \cup B|$?

13. The formula you just found is the easiest case of the inclusion-exclusion principle.
• Why do you think it has this name?
• What happens to this formula if *A* and *B* are disjoint?

14. In a class of 50 students, there are 30 girls, and there are 35 students with dark hair. Use the inclusion-exclusion principle to explain why there must be at least 15 girls with dark hair in the class.

15. Use the inclusion-exclusion principle to find the number of positive integers less than or equal to 100 which are divisible by 3 or 7. Do NOT write down the integers from 1 to 100 to figure this out.

16. The inclusion-exclusion principle gets more complicated when there are more than two sets to consider. But for three sets, it is not too bad. Draw a generic Venn diagram of three sets A, B and C and use it to figure out a formula for $|A \cup B \cup C|$.

Hint: you will include, then exclude, and then include again!

17. Find the number of integers in the set $S = \{1, 2, 3, \ldots, 6300\}$ which are divisible by 3 or 4 or 5.

In general, the inclusion-exclusion principle tells us that in order to find $|A_1 \cup A_2 \cup \cdots \cup A_k|$, the number of elements in the union of the k sets $A_1, A_2, \ldots, A_k$, we have to:

- add the number of elements of each of these *k* sets,
- subtract the number of elements of each intersection of two of the sets,
- add the number of elements of each intersection of three of the sets,
- and so on until we add (if *k* is odd) or subtract (if *k* is even) the number of elements in the intersection of all *k* sets.

For example, if $k = 4$, we get

$$\begin{aligned}|A \cup B \cup C \cup D| = {} & |A| + |B| + |C| + |D| \\ & -|A \cap B| - |A \cap C| - |A \cap D| - |B \cap C| - |B \cap D| - |C \cap D| \\ & +|A \cap B \cap C| + |A \cap B \cap D| + |A \cap C \cap D| + |B \cap C \cap D| \\ & -|A \cap B \cap C \cap D|.\end{aligned}$$

18. Explain what happens to this formula if *D* is disjoint from the other three sets. What happens if the four sets are pairwise disjoint? You should see that the addition principle is nothing more than a very special case of the inclusion-exclusion principle.

Complements

Sometimes we are asked to find the number of elements from a set that possess, or do not possess, a certain property or a list of properties. In such cases, it is a good idea to consider whether it is easier to count the "complement", that is to say the elements that we are not asked to count. If *A* is the subset of *S* consisting of those elements satisfying some property, then its complement is the subset *B* of *S* consisting of those elements that do not satisfy the property. Then clearly *S* is the disjoint union of *A* and *B*, so that by the addition principle, $|S| = |A| + |B|$.

19. Find the number of integers in the set $S = \{1, 2, 3, \ldots, 6300\}$ which are divisible by none of 3, 4 or 5.

Ordered and Unordered Selections (Permutations and Combinations)

20. If we have to make an ordered selection (that is, the order matters) of k objects with no repetitions from a set of n objects, explain why the number of ways of doing this is $\frac{n!}{(n-k)!}$. In particular, the number of permutations of the elements of a set consisting of n elements is $n!$.
Note: by definition, 0!=1.

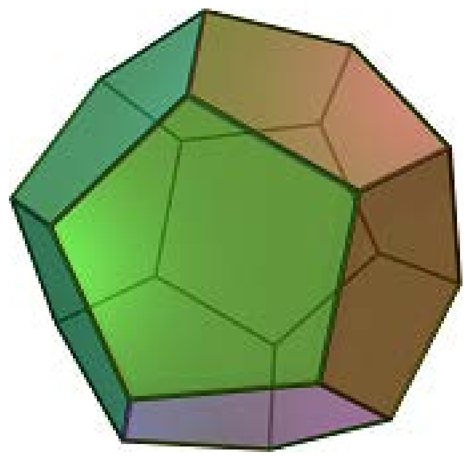
21. Prove that the number of ways of choosing k objects, with no repetitions and where the order does not matter, from a set of n objects is $\binom{n}{k} = \frac{n!}{k!(n-k)!}$. Also explain in two different ways why $\binom{n}{k} = \binom{n}{n-k}$.

22. Prove the Binomial Theorem: $(x+y)^n = \sum_{k=0}^{n} \binom{n}{k} x^k y^{n-k}$. Then explain in two ways why $\binom{n}{0} + \binom{n}{1} + \binom{n}{2} + \cdots + \binom{n}{n} = 2^n$.

Over-Counting and Compensating

23. How many diagonals does a (convex) n-gon have?

24. The dodecahedron is one of the five "platonic solids". It has 12 faces, each of which is a regular pentagon. Three faces meet at each vertex and two faces meet along each edge. Here is a picture:



Use the method of over-counting and compensating to figure out how many edges and how many vertices it has.

The Pigeonhole Principle

The pigeonhole principle says that if you put more than n pigeons into n holes (in other words: there are more pigeons than holes), then at least one hole will contain more than one pigeon.

More generally, if you put more than kn pigeons into n holes, then at least one hole will contain more than k pigeons.

Here are three problems from the Berkeley Math Circle book:

25. An ice cream shop sells four flavors of ice cream. Seven friends show up, and each of them orders a cone with two different flavors. Prove that at least two of them ordered the same combination of flavors.

26. 15 chairs are placed around a circular table. On the table are name cards for 15 guests. After the guests sit down, they realize that none of them is sitting in front of his or her own card. Prove that the table can be rotated so that at least two guests are simultaneously correctly seated.

27. Consider five points in the interior of a square of side 1. Show that two of these points are no more than $1/\sqrt{2}$ apart.