Part B: A Sampler of Exercises

Below is a list of problems for you to try. They vary in difficulty. Most of them (but not all of them) are about counting or have some connection to counting. Have fun!

1. Ana, Ben, Chris, Dave, Esther and Frank get on an elevator on the ground floor of Evans Hall. Each of them pushes the floor number where they will get off, which is a number between 1 and 10. Of all the ways in which this can happen, how many result in at least two people getting off the elevator on the same floor?

2. True or false: at a party of six people, there must be three people such that either any two of them know each other or any two of them don't know each other. How about the same question for a party of five people?

3. Prove that there are no polyhedra with no two faces with the same number of sides. That is, prove that a polyhedron must have two faces with the same number of sides.

4. If you are dealt five cards at random from a standard deck of 52 cards, what is the probability that you have "poker" (that is, four of a kind: the four aces, or the four 2s, …, or the four kings)? Find an expression for this probability. You do not have to evaluate it.

5. If you are dealt seven cards at random from a standard deck of 52 cards (recall that there are four suits, each having 13 cards. The suits are clubs, diamonds, hearts and spades), what is the probability that you will have at least one club? Find an expression for this probability. You do not have to evaluate it.

6. There are five couples at a party. One of these ten people is a mathematician who asked each other person how many people they had shaken hands with at the party. She received a different answer from each person ranging from 0 through 8. Partners did not shake hands with each other. How many people did the mathematician's partner shake hands with?

7. Ana, Ben and Chris spent the afternoon playing ping-pong. They played several matches, with the rule that "winner stays, loser goes out". At the end, Ana had played 10 matches, Ben 15, and Chris 17. Who lost the second match?

8. 27 small cubes are used to form a large cube. Is it possible to paint the faces of the small cubes with three colors (red, green, and yellow) in such a way that:
- in one way of assembling the large cube, all six faces are red,
- in another way, all six faces are green, and
- in still another way, all six faces are yellow?

In general, if you have n^3 unit cubes, can you paint their faces using n colors so that for each color, you can put together a large, $n \times n \times n$ cube in a way that all its faces are that color?

9. Given five points on a sphere, show that some four of them must lie on a closed hemisphere.

10. Can a 6x6 square grid be tiled with the "L" shaped tetromino? (reflections and rotations of the tetromino are allowed)

11. On a 5×5 board, place five wolves (they move like queens) and three sheep so that each sheep is safe from all the wolves.

12. Suppose you have a rectangular $m \times n$ board (like a chessboard if $m = n = 8$) with mn squares. Each square has an arrow which is pointing up, left, down, or right. Choose any initial square to start traveling on the board, following these rules:

(a) Move to the adjacent square indicated by the arrow (up, left, down or right).
(b) When you arrive at each new square, turn the arrow of the previous square (the square you just came from) 90° clockwise.
(c) If at any point you are on a square on the edge of the board, and the arrow points outside, then you leave the board and the trip is over.

Is it possible to choose a board (that is, its dimensions m and n), an initial configuration of the arrows in its squares, and an initial starting square so that you can travel indefinitely in the board, never leaving it? Or must you always eventually leave the board?

13. One day, the pirates of a ship found a hidden treasure consisting of gold coins. They shared them equally, so that each pirate had the same number of coins. But then, back in their ship, at different moments of the day, some pirate (not necessarily the same one every time) would give away some of his coins, giving every other pirate the same number of coins. At some point that evening, there was one pirate who had 19 coins and another pirate who had 150 coins.
How many pirates were there aboard the ship?

14. (a) Let $S = \{1, 2, 3, ..., 20\}$. Suppose that 12 distinct numbers are chosen among the elements of S. Let A denote the subset of S consisting of these 12 numbers. Prove that there must be two distinct numbers in A whose sum is also in A.

(b) Show that the above result is "best possible", in the sense that if A has fewer than 12 elements, then the conclusion is false.