COLORING PROBLEM

IGOR GANICHEV

BMC, April 24^{th}, 2019

1. Problems

Problem 1.1. Can one cover a chess board without cells $a1$ and $h8$ with dominos?

Problem 1.2. Can one cover a 4×5 rectangle with 5 tetris figures?

Problem 1.3. A castle consists of 35 rooms arranged into 5×7 rectangle. Each wall between two rooms (an edge in the 5×7 rectangle) contains a door. Is it possible to visit all rooms in the castle exactly once without exiting the castle and without entering the central room?

Problem 1.4. Show that a 50×50 board cannot be covered with T-shaped tetrominoes?

Problem 1.5. A triangular castle consists of 100 identical triangular rooms. The rooms are arranged in rows containing $1, 3, 5, \ldots, 19$ triangles. Each wall between two triangles contains a door. What is the largest number of rooms that a visitor can visit without entering the same room more than once? As an illustration, the left picture in figure 1 shows a path visiting 91 rooms.

Problem 1.6. Given an equilateral triangle with edge 5, how many diamonds (two equilateral triangles put together) can one cut from it? For simplicity, assume that cuts can be made along the "triangular lattice" only. See center picture in figure 1.

Problem 1.7. What is the smallest number of rectangles that the shape in the right side of figure 1 can be cut? Cuts to be done along the lattice lines.

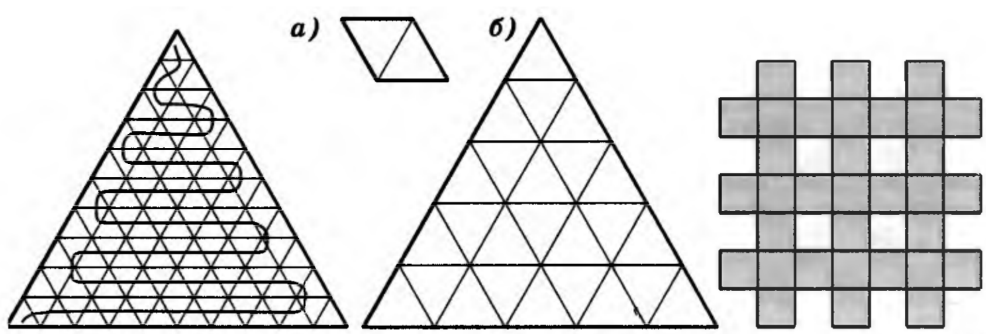

Figure 1. Illustrations for questions 1.5, 1.6, and 1.7

.

Problem 1.8. 25 bugs were sitting on a 5×5 board. Each bug on a different cell. At one time, each bug flew to a neighboring cell (i.e. a cell sharing an edge). Show that at least one cell ended up being empty.

Problem 1.9. A 100×100 square is separated into rectangles by several lines drawn along the lattice lines. The rectangles are colored red and blue following a checkers pattern. Show that the number of blues cells is even.

Problem 1.10. Can one cover the chess board with 15 vertical and 17 horizontal dominoes?

Problem 1.11. Prove that a 4×11 rectangle cannot be tiled by L-shaped tetrominoes.

Problem 1.12. A chess board is covered with dominoes. Pick one diagonal of the board. This diagonal separates of the board into two regions A and B. Some dominoes covering a cell of the diagonal also cover a cell of region A and some of region B. Show that there are 4 dominoes of each type.

Problem 1.13. Can a 5×7 board be covered by L-trominos, not crossing its boundary, in several layers, so that each square of the board is covered by the same number of trominos?

Problem 1.14. Let n, a, b be positive integers such that it is possible to tile an $a \times b$ rectangle using $1 \times n$ rectangles (rotations allowed). Prove that $n|a$ or $n|b$.

Problem 1.15. A 6×6 board is tiled by dominoes. Show that there is a line that cuts the board into two parts without cutting any domino.

Problem 1.16. What is the maximum number of dominoes which can be placed on an 8×9 board if six are already placed as shown in Figure 3?

Problem 1.17. A computer screen shows a 98×98 chessboard, colored in the usual way. One can select with a mouse any rectangle with sides on the lines of the chessboard and click the mouse button: as a result, the colors in the selected rectangle switch (black becomes white, white becomes black). Determine the minimum number of mouse clicks needed to make the chessboard all one color.

Problem 1.18. There are n lamps arranged (evenly spaced) in a circle. Initially, one of them is turned *on*, and the rest are *off*. It is permitted to choose any regular polygon whose vertices are lamps which are all in the same state (either all *on* or all *off*), and change all of their states simultaneously. For which positive integers n is it possible to turn all the lamps *off* after a finite number of such operations?

Problem 1.19. (IMO 2004) Define a hook to be a figure made up of six unit squares as shown in Figure 2, or any of the figures obtained by applying rotations and reflections to this figure. Determine all $m \times n$ rectangles that can be tiled with hooks?[1]

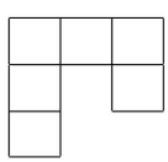

FIGURE 2. The hook shape.

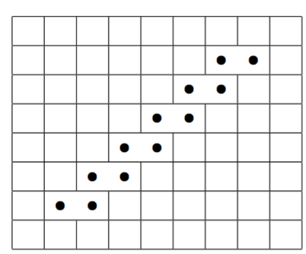

FIGURE 3. Pre-placed dominoes

[1]For an analysis of a large number of polyomino tiles in terms of which rectangles each can tile, see `http://www.cflmath.com/Polyomino/index.html`