Sequences

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Instructions: This text is for reading *after* the class. During class, pay attention. If you get bored, try some of the harder problems in the mixed problems section, or have a go at the unsolved problem in Section 4.

We consider sequences of numbers, e.g.

$$1, 2, 4, 8, 16, \dots$$
$$1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \dots$$
$$1, 1, 2, 3, 5, 8, 13, \dots$$

Generally we write $a_0, a_1, a_2, \dots$ or (a_n) for a sequence.[2]
For the first two sequences there are **closed formulas**: $a_n = 2^n$ and $a_n = \frac{1}{n+1}$ for all n, respectively.
The third sequence is built from the law $a_n = a_{n-1} + a_{n-2}$ (for $n \geq 2$). This is a **recurrence relation**, i.e. a rule by which a_n can be foand from

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[2]One also writes $(a_n)_{n\in\mathbb{N}_0}$. In this text n will always start at $n = 0$.

previous sequence elements. We also speak of a recursively defined sequence. For a recurrence relation to determine a sequence uniquely you must specify **initial values**, in this case $a_0 = 1$, $a_1 = 1$.

1 Recursively defined sequences

There are many interesting problems connected with recursively defined sequences. For example:

1. Can we find a closed formula?

2. How do the a_n behave for large n? Do they approach some number, or infinity, as $n \to \infty$? (We then also say that this is the limit, and that a_n tends to this limit, and write $a_n \to x$ if the limit is x.)[3] If yes, how quickly? If they tend to infinity, then how fast (for example exponentially, polynomially)?

Problem 1. *Find a closed formula for the sequences defined by:*
$a_n = a_{n-1} + 2, a_0 = 1$; $b_n = 3b_{n-1}$, $b_0 = 1$; $c_n = 2c_{n-1} + 1$, $c_0 = 2$.

Often it is difficult or impossible to find a closed formula. Sometimes you can still find the behavior as $n \to \infty$:

Example (Finding a limit from a recurrence relation)**.**

$$a_{n+1} = \frac{1}{2}\left(a_n + \frac{1}{a_n}\right), \; a_0 = 2$$

The first terms are $2, \frac{5}{4}, \frac{41}{40}$. *This leads us to* ***conjecture****:*

1. *All* a_n *are bigger than* 1.

2. *The sequence is* ***decreasing****, i.e.* $a_{n+1} \leq a_n$ *for all* n.

3. *The terms of the sequence approach the limit* 1.

The first two claims are easy to check. (In 1. use $x + \frac{1}{x} \geq 2$ *for all* $x > 0$*, with equality if and only if* $x = 1$*.)*
Then the third claim follows from the first two. This works in two steps.

[3] We will not treat limits formally (that's done in an analysis course). Here are some simple examples for intuition: $a_n = \frac{1}{n}$, that is $1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \ldots$ tends to 0, $a_n = n$ tends to ∞ and the sequence $0, 1, 0, 1, 0, 1, \ldots$ has no limit.

First, the following theorem holds. It should be intuitively clear (and is proved rigorously in any analysis class):

Theorem: *A decreasing sequence which is bounded from below has a limit. (Of course there is a similar fact for increasing and bounded above.)*

In the second step, we find the limit: *Call the limit x. How do we calculate it? Since x is the limit of the a_n, it is also the limit of the a_{n+1}, so we get from the recurrence relation for $n \to \infty$:*

$$\begin{array}{ccc} a_{n+1} & = & \frac{1}{2}\left(a_n + \frac{1}{a_n}\right) \\ \downarrow & & \downarrow \\ x & = & \frac{1}{2}\left(x + \frac{1}{x}\right) \end{array}$$

This is the ***fixed point equation****. A short calculation then gives $x = 1$ (or $x = -1$, but x must be positive).*
Therefore, the limit of the a_n is 1.

Problem 2. *Investigate how the sequence defined by $a_{n+1} = \frac{1}{2}\left(a_n + \frac{2}{a_n}\right)$, $a_0 = 5$ behaves as $n \to \infty$.*

Use a calculator to find the first 6 terms of this sequence. You will observe that it approaches 1.4142... very quickly, which is $\sqrt{2}$. Why?

Problem 3. *Investigate how fast the sequence in Problem 2 approaches its limit.*

Problem 4. *Investiate the behavior of the sequence defined by $a_{n+1} = 1 - a_n$, $a_0 = 0$ as $n \to \infty$. What is the solution of the fixed point equation?*

We see: it is not sufficient to just solve the fixed point equation.

Linear recurrence relations

We want to find a closed formula for the Fibonacci numbers, which are defined by the recurrence relation

$$F_n = F_{n-1} + F_{n-2}\,, \qquad F_0 = F_1 = 1$$

The first few are $1, 1, 2, 3, 5, 8, \ldots$.
Try to guess a formula! It seems impossible.
There are (at least) two systematic methods to find a closed formula:

1. By a power ansatz.[4]

2. Using generating functions.

The power ansatz is simpler but generating functions are more powerful: they can be used for more problems, as we will see!

Solving the Fibonacci recurrence relation by a power ansatz

Step 1: At first, forget about the initial condition. Just try to find a solution of the recurrence relation $a_n = a_{n-1} + a_{n-2}$ alone. Try $a_n = x^n$.
So we must have, for all n: $x^n = x^{n-1} + x^{n-2}$.
This is equivalent[5] to $x^2 = x + 1$. This equation has two solutions[6] $x = \alpha$ and $x = \beta$:

$$\alpha = \frac{1+\sqrt{5}}{2}, \quad \beta = \frac{1-\sqrt{5}}{2} \tag{1}$$

So both the sequences (α^n) and (β^n) satisfy the FIBONACCI recurrence relation. But not the initial condition.

Step 2: How can we satisfy the initial condition?
Important observation: If A, B are any numbers then the sequence $a_n = A\alpha^n + B\beta^n$ also satisfies the recurrence relation.
So we just need to find A, B in such a way that the initial conditions are satisfied. Taking $n = 0$ and $n = 1$ we get $1 = A + B, 1 = A\alpha + B\beta$. If that is satisfied then it follows that $F_n = a_n$.
A short calculation yields $A = \frac{\alpha}{\sqrt{5}}$, $B = -\frac{\beta}{\sqrt{5}}$, thus:

$$\boxed{F_n = \frac{1}{\sqrt{5}}\left[\left(\frac{1+\sqrt{5}}{2}\right)^{n+1} - \left(\frac{1-\sqrt{5}}{2}\right)^{n+1}\right]} \tag{2}$$

This formula seems crazy! The irrational number $\sqrt{5}$ appears at three places, but we know that all the F_n are integers. How does that fit together? Try multiplying out the powers for $n = 1, 2, 3$ and see what happens![7]

[4] Ansatz = a guessed form of a solution, which has some indeterminates that can be found by plugging in the ansatz into the equation. This German word is also used in mathematical English.

[5] if $x \neq 0$; of course $x = 0$ gives a solution but a boring one.

[6] Recall how to solve this quadratic equation: $x^2 = x + 1 \iff x^2 - x = 1$, now complete the square: $\iff x^2 - 2 \cdot \frac{1}{2}x + \left(\frac{1}{2}\right)^2 = \left(\frac{1}{2}\right)^2 + 1 \iff \left(x - \frac{1}{2}\right)^2 = \frac{5}{4} \iff x - \frac{1}{2} = \pm\sqrt{\frac{5}{4}} = \pm\frac{\sqrt{5}}{2} \iff x = \frac{1\pm\sqrt{5}}{2}$

[7] If you know the general binomial formula explain why the formula always yields a rational number.

Problem 5. *Solve the recurrence relation* $a_n = 7a_{n-1} - 12a_{n-2}$*,* $a_0 = 2$*,* $a_1 = 7$*.*

Problem 6. *Solve the recurrence relation* $a_n = 2a_{n-1} - a_{n-2}$ *(a) with initial conditions* $a_0 = a_1 = 1$*, (b) with initial conditions* $a_0 = 0$*,* $a_1 = 1$*.*

In this problem you see that the power ansatz does not always work. The reason is that the equation $x^2 - 2x + 1 = 0$ has only one solution $x = 1$. This gives the solution in (a) but not the solution in (b).

Problem 7. *Prove that any solution of the recurrence relation* $a_n = \sqrt{2}a_{n-1} - a_{n-2}$ *has period* 8*, that is,* $a_{n+8} = a_n$ *for all* n*.*

One way to do this is using complex numbers and the power series ansatz: the solutions of $x^2 - \sqrt{2}x + 1$ are $x_\pm = \frac{1}{\sqrt{2}}(1 \pm i)$. Squaring yields $\pm i$, and squaring two more times you get 1. [8]

Advanced reading: Some general theory for linear recurrence relations:
A **linear recurrence relation** is one of the form

$$a_n = c_{k-1}a_{n-1} + \cdots + c_0 a_{n-k}\,, \tag{3}$$

where $k \in \mathbb{N}$ and $c_0, \dots, c_{k-1} \in \mathbb{R}$ are given. The integer k is called the length of the recurrence. The FIBONACCI recurrence is linear with length 2.
As initial conditions you need the k numbers $a_0, \dots, a_{k-1}$.
Let us try the power ansatz:

Step 1: We look for sequences $a_n = x^n$ satisfying the recurrence relation. This means $x^n = c_{k-1}x^{n-1} + \cdots + c_0 x^{n-k}$. Dividing by x^{n-k} and reordering we see that this is equivalent to

$$p(x) = 0, \text{where } p(x) := x^k - c_{k-1}x^{k-1} - \cdots - c_0\,. \tag{4}$$

That is: $a_n = x^n$ satisfies the recurrence relation if and only if x is a zero of the polynomial $p(x)$. This polynomial is called the **characteristic polynomial** of the recurrence relation.
Background knowledge: a polynomial of degree k has at most k distinct zeroes.

Step 2: How do we find a sequence satisfying, in addition, the initial conditions? Suppose p has precisely k zeroes $x_1, \dots, x_k$. We look for numbers $A_1, \dots, A_k$ so that $a_n = A_1 x_1^n + \cdots + A_k x_k^n$ satisfies the initial condition. That is, for given numbers $a_0, \dots, a_{k-1}$ we must have

$$\begin{array}{llcccc} a_0 & = A_1 & + & \dots & + & A_k \\ a_1 & = A_1 x_1 & + & \dots & + & A_k x_k \\ \vdots & & & & & \\ a_{k-1} & = A_1 x_1^{k-1} & + & \dots & + & A_k x_k^{k-1} \end{array}$$

[8] Another quick way to see this is to use the polar representation of complex numbers and Euler's formula: $\frac{1}{\sqrt{2}}(1 \pm i) = e^{\pm i\pi/4}$ implies $x_\pm^8 = e^{\pm 8 \cdot i\pi/4} = e^{\pm 2\pi i} = 1$.

This is a system of k linear equations for the k unknowns $A_1, \dots, A_k$. One can show that it always has a non-zero solution.[9]
If p has fewer than k distinct zeroes then we get k equations for less than k unknowns, and this is not solvable for arbitrary given $a_0, \dots, a_{k-1}$. Compare Problem 6.
Note that the x_i and the A_i can be complex numbers. They may be non-real even if the recurrence relation and the initial conditions are real. See Problem 7 for an example.

Result: The linear recurrence relation (3) with arbitrary initial conditions can be solved by the power ansatz if and only if the characteristic polynomial $p(x)$ in (4) has k distinct roots.

What do you do if p has fewer roots? See below!

2 Generating functions

Generating functions are an amazingly powerful tool for analyzing sequences.

Definition 2.1. *The* ***generating function*** *of a sequence* $a_0, a_1, \dots$ *is the function*

$$f(x) = a_0 + a_1 x + a_2 x^2 + \dots$$

that is, the power series with coefficients $a_0, a_1, \dots$.[10]

Sometimes you can simplify the infinite sum. Fundamental example: $a_0 = a_1 = \dots = 1$.

$$\boxed{\textbf{Geometric series: } 1 + x + x^2 + x^3 + \dots = \frac{1}{1-x}} \tag{5}$$

for $|x| < 1$. This follows from the formula for the geometric sum

$$1 + x + x^2 + \dots + x^n = \frac{1 - x^{n+1}}{1-x}$$

by letting $n \to \infty$ since $x^{n+1} \to 0$ if $|x| < 1$.[11]
Squaring (5) and multiplying out we get $1 + 2x + 3x^2 + \dots = \frac{1}{(1-x)^2}$. You get the same result by differentiating (5).

[9] The coefficient matrix is the so-called *Vandermonde* matrix, it is always invertible if all x_i are distinct.

[10] If you want to know the domain of definition of f then you should investigate for which x the infinite sum converges. However, for many problems (in particular for these notes) this is irrelevant. This statement can be justfied rigorously by considering all series as *formal power series.*

[11] Another, formal proof: $(1 + x + x^2 + \dots)(1 - x) = 1 - x + x - x^2 + x^2 - x^3 \dots = 1$.

Problem 8. *Find simple expressions for the generating functions of the sequences*
$a_n = 2^n$, $b_n = n$, $c_n = n2^n$, $d_n = n^2$, $(e_n) = (1, 0, 1, 0, 1, 0, \dots)$

Solution of the Fibonacci recurrence relation via generating functions

Using $F_n = F_{n-1} + F_{n-2}$, $F_0 = F_1 = 1$ we get

$$\begin{aligned} f(x) &= F_0 + F_1 x + F_2 x^2 + F_3 x^3 \dots \\ &= 1 \;+\; \;\; x + (F_1 + F_0)x^2 + (F_2 + F_1)x^3 + \dots \\ &= 1 + (x + F_1 x^2 + F_2 x^3 + \dots) + (F_0 x^2 + F_1 x^3 + \dots) \\ &= 1 + x f(x) + x^2 f(x) \end{aligned}$$

hence

$$f(x) = \frac{1}{1 - x - x^2} = -\frac{1}{x^2 + x - 1}$$

Now the trick is to expand this in a power series by a different route:

Step 1: Partial fractions. The zeroes of the polynomial $x^2 + x - 1$ are $a = \frac{-1+\sqrt{5}}{2}$ and $b = \frac{-1-\sqrt{5}}{2}$, so we have $x^2 + x - 1 = (x - a)(x - b)$. We now look for numbers C, D satisfying

$$\frac{1}{x^2 + x - 1} = \frac{C}{x - a} + \frac{D}{x - b}\,.$$

Multiplying by $x^2 + x - 1 = (x - a)(x - b)$ we get

$$1 = C(x - b) + D(x - a) = (C + D)x + (-Cb - Da)$$

and by comparing coefficients of like powers of x we get $0 = C + D$, $1 = -Cb - Da$. After a short calculation we find $C = \frac{1}{\sqrt{5}} = -D$, so

$$f(x) = -\frac{1}{\sqrt{5}} \left(\frac{1}{x - a} - \frac{1}{x - b} \right) \tag{6}$$

Step 2: Using the geometric series we now obtain

$$\frac{1}{x - a} = \frac{1}{a} \frac{1}{\frac{x}{a} - 1} = -\frac{1}{a} \frac{1}{1 - \frac{x}{a}} = -\frac{1}{a} \left(1 + \frac{x}{a} + \frac{x^2}{a^2} + \dots \right)$$

The coefficient of x^n is $-a^{-n-1}$. The result for $\frac{1}{x-b}$ is analogous, and using (6) we get that the coefficient of x^n in $f(x)$ is

$$\frac{1}{\sqrt{5}}\left(a^{-n-1} - b^{-n-1}\right)$$

is. Now this coefficent is also equal to F_n, by the definition of $f(x)$. So this is the formula for F_n that we were looking for. (It looks different than (2) but is actually the same because of $a = \frac{1}{\alpha}$, $b = \frac{1}{\beta}$.)[12]

Linear recurrences for which the power ansatz fails

Using generating functions you can also find the 'missing' solution in Problem 6 which we could not find using the power ansatz:

Problem 9. *Find a closed formula for the solution of $a_n = 2a_{n-1} - a_{n-2}$, $a_0 = 0$, $a_1 = 1$ using generating functions.*

Advanced reading: General theory of linear recurrence relations, part II:
How do we solve a general linear recurrence relation (3) using generating functions? Let $p(x)$ be the characteristic polynomial as in (4). The generating function for the a_n has the form $f(x) = \frac{r(x)}{q(x)}$, where $q(x) = x^k p(\frac{1}{x})$ is the polynomial reciprocal to p and where r is a polynomial of degree $< k$ which is determined using the initial conditions. The partial fractions decomposition of f is a sum of terms of the form $\frac{A}{(x-z_i)^l}$ where z_i are the zeroes of q and where $l = 0, \dots, d_i - 1$, with d_i denoting the multiplicity of the zero z_i. Now you can expand the function $\frac{1}{(1-x)^l}$ as power series with coefficients $n(n-1)\dots(n-l+1)$. Note that the latter is a polynomial in n of degree k. Proceeding as in the FIBONACCI example we get:

Result: Suppose that the characteristic polynomial (4) of the linear recurrence relation (3) has the zeroes $x_1, \dots, x_m$ with multiplicities $d_1, \dots, d_m$, respectively. Then the general solution of the recurrence relation is a sum of terms of the form $A\,n^l x_i^n$, $0 \le l < d_i$, $i = 1, \dots, m$.

Note that p having less than k zeroes is equivalent to p having multiple zeroes (i.e. some d_i is bigger than 1).

In short: *If p has multiple zeroes than the power ansatz needs to be extended to include terms of the form $n^l x^n$, where l is less than the multiplicity of x as a zero of p.*

[12] It is not a coincidence that $a = \frac{1}{\alpha}$, $b = \frac{1}{\beta}$: α, β are the zeroes of $p(x) = x^2 - x - 1$, the characteristic polynomial of the FIBONACCI recurrence. The polynomial $1 - x - x^2$ which occured as denominator of $f(x)$ is *reciprocal* to $p(x)$ in the sense that the order of coefficients is reversed, or equivalently $1 - x - x^2 = x^2 p(\frac{1}{x})$. Therefore its zeroes must be $\frac{1}{\alpha}$, $\frac{1}{\beta}$. All of this generalizes to any linear recurrence relation. Check it!

3 Partition numbers

Let p_n be the number of **partitions** of $n \in \mathbb{N}$, i.e. of ways to represent n as a sum of natural numbers, ignoring order. For example, the partitions of $n = 4$ are

$$4,\ 1+3,\ 2+2,\ 1+1+2,\ 1+1+1+1,$$

so $p_4 = 5$.[13] We also define $p_0 = 1$. The partition numbers for $0 \leq n \leq 10$ are $1, 1, 2, 3, 5, 7, 11, 15, 22, 30, 42$.
There is no obvious closed formula or recurrence relation. But:

Problem 10. *Show that the generating function of the partition numbers is*

$$p(x) = \frac{1}{1-x} \cdot \frac{1}{1-x^2} \cdot \frac{1}{1-x^3} \cdots$$

Let

$$\begin{aligned} o_n &= \text{ the number of partitions of } n \text{ with } \textit{odd} \text{ summands} \\ d_n &= \text{ the number of partitions of } n \text{ with } \textit{distinct} \text{ summands} \end{aligned}$$

Problem 11. *Find o_n, d_n for $n = 1, \ldots, 8$. Conjecture?*

Using generating functions it is not difficult to prove the conjecture (again we set $o_0 := 1$, $d_0 := 1$):

Problem 12. *Show that the sequences (o_n), (d_n) have generating functions*

$$\begin{aligned} o(x) &= \frac{1}{1-x} \cdot \frac{1}{1-x^3} \cdot \frac{1}{1-x^5} \cdots \\ d(x) &= (1+x) \cdot (1+x^2) \cdot (1+x^3) \cdots \end{aligned}$$

Problem 13. *Prove that $o(x) = d(x)$.*

So this implies that $o_n = d_n$ for all n. It is not easy to prove this directly from the definition. Try it!
Partition numbers have many more surprising properties. For example one can show that

$$p_n \sim \frac{1}{4n\sqrt{3}} e^{\sqrt{n}\pi\sqrt{2/3}}$$

[13] It is customary to count a sum with just one term also as a partition. In this way all formulas are much nicer than they would be otherwise.

where $\sim$ means that the ratio of the left and right hand sides tends to 1 as $n \to \infty$.[14] Also, the partition numbers satisfy a highly non-trivial (and hard to find!) recurrence relation:

$$p_n = p_{n-1} + p_{n-2} - p_{n-5} - p_{n-7} + p_{n-12} + p_{n-15} - \dots$$

(Euler's pentagonal number theorem)

4 An unsolved problem: The Collatz problem

Define a sequence as follows: Pick a natural number a_0 and then let

$$a_{n+1} = \begin{cases} a_n/2 & \text{if } a_n \text{ is even} \\ 3a_n + 1 & \text{if } a_n \text{ is odd} \end{cases}$$

Let's try some examples:

- $a_0 = 1$ yields $1, 4, 2, 1, 4, 2, 1, \dots$
- $a_0 = 3$ yields $3, 10, 5, 16, 8, 4, 2, 1, 4, 2, 1, \dots$

If you try some more values of a_0, you will find that sooner or later you will always get 1 (and then $4, 2, 1, \dots$). It is an **unsolved problem** to prove (or disprove) that this is true for *all* initial values a_0.

5 Mixed problems

Problem 14. *A sequence begins* $1, 2, 4$. *What's the next term?*

Problem 15. *Let* $a_{n+1} = \frac{1}{2}\left(a_n + \frac{1}{a_n}\right)$, $a_0 = 2$. *Find a closed formula for* a_n.

Problem 16. *Let* $x > 0$. *How can you calculate* $\sqrt{x}$ *quickly to many digits, using only the basic arithmetic operations?*

Problem 17. *Let* $a_0 = a_1 = 1$ *and* $a_n = \sqrt{a_{n-1}} + \sqrt{a_{n-2}}$ *for* $n \geq 2$. *What happens for* $n \to \infty$*?*

Problem 18. *Find* $\sqrt{6 + \sqrt{6 + \sqrt{6 + \dots}}}$. *What does this expression actually mean?*

[14] To prove this you need complex analysis. See the book Tom Apostol: Introduction to Analytic Number Theory.

Problem 19. *Let* $a_0 = 1$, $a_{n+1} = a_n + \frac{1}{a_n}$. *Is the sequence* (a_n) *bounded?*

Problem 20. *Let* $\alpha = \frac{1+\sqrt{5}}{2}$ *be the golden ratio. Find* $\frac{1}{\sqrt{5}}\alpha^{12}$ *to two digits after the decimal point without using a calculator.*

Problem 21. *In how many ways can you tile a* $2 \times n$ *rectangle using* 1×2 *dominoes?*

Problem 22. *Show that* p_n, *the number of partitions of* n, *satisfies* $p_n \geq 2^{\lfloor \sqrt{n} \rfloor}$ *for* $n \geq 2$.

6 Hints

Hint 1. For c_n: Add 1, then the recurrence relation changes to $c_n + 1 = 2(c_{n-1} + 1)$. What does this mean for $d_n = c_n + 1$?
Hint 2. Is (a_n) decreasing? The arithmetic-geometric mean inequality (AGM) is useful: $\sqrt{xy} \leq \frac{x+y}{2}$ for $x, y \geq 0$, with equality if and only if $x = y$.
Hint 3. Consider $b_n = a_n - \sqrt{2}$ and estimate b_{n+1} in terms of b_n.
Hint 6. First calculate some terms of the sequence.
Hint 8. For d_n differentiate (5) twice.
Hint 10. Write each factor as a geometric series, then multiply out. In which ways can the term x^n appear?
Hint 13. $1 + x^k = \frac{1-x^{2k}}{1-x^k}$
Hint 15. $a_1 = \frac{1}{2}(2 + \frac{1}{2}) = \frac{5}{4}$, $a_2 = \frac{1}{2}\left(\frac{5}{4} + \frac{4}{5}\right) = \frac{1}{2}\frac{4^2+5^2}{4 \cdot 5} = \frac{41}{40}$. Do you see a pattern?
Hint 17. First solve the fixed point equation.
Hint 18. Set $a_0 = 0$, $a_n = \sqrt{6 + a_{n-1}}$ and find the limit for $n \to \infty$.
Hint 20. $\beta = \frac{-1+\sqrt{5}}{2} = -0.618\ldots$
Hint 21. Let a_n be this number. Find a recurrence relation.
Hint 22. Use that $2^{\lfloor \sqrt{n} \rfloor}$ is the number of subsets of $\{1, 2, \ldots, \lfloor \sqrt{n} \rfloor\}$.

7 Solutions

Solution 1. $a_n = 2n + 1$, $b_n = 3^n$, $c_n = 3 \cdot 2^n - 1$
Solution 2. $a_{n+1} \leq a_n \iff \frac{1}{2}(a_n + \frac{2}{a_n}) \leq a_n \iff \frac{2}{a_n} \leq a_n \iff 2 \leq a_n^2$. So we need to check whether $a_n^2 \geq 2$ for all n. By the AGM inequality $a_n = \frac{1}{2}(a_{n-1} + \frac{2}{a_{n-1}}) \geq \sqrt{a_{n-1}\frac{2}{a_{n-1}}} = \sqrt{2}$, so this holds for $n \geq 1$, and since it is true for $a_0 = 5$, the sequence (a_n) decreases for all n. Since it is

bounded below, the sequence converges to a limit x. The limit must satisfy $x = \frac{1}{2}(x + \frac{2}{x})$, so $x = \sqrt{2}$.
Remark: The initial value is irrelevant, as long as it's positive. If $a_0 < \sqrt{2}$ then (a_n) decreases only starting at $n \geq 1$.
Solution 3. For $b_n = a_n - \sqrt{2}$ we have $b_{n+1} = a_{n+1} - \sqrt{2} = \frac{1}{2}(a_n + \frac{2}{a_n}) - \sqrt{2} = \frac{a_n^2 + 2 - 2\sqrt{2}a_n}{2a_n} = \frac{(a_n - \sqrt{2})^2}{2a_n} < \frac{1}{2}b_n^2$ for all n using $a_n > 1$. So the deviation of a_n from the limit $\sqrt{2}$ gets at least squared and halved in each step. So the number of correct digits after the decimal point roughly doubles in each step.
Solution 4. $0, 1, 0, 1, \ldots$ has no limit as $n \to \infty$ (although the fixed point equation $x = 1 - x$ has the solution $x = \frac{1}{2}$).
Solution 5. $a_n = 3^n + 4^n$
Solution 6. (a) $a_n = 1$ for all n. (b) $a_n = n$.
Solution 7. Let $a = a_n$, $b = a_{n+1}$, then $a_{n+2} = \sqrt{2}b - a$, $a_{n+3} = \sqrt{2}a_{n+2} - a_{n+1} = 2b - \sqrt{2}a - b = b - \sqrt{2}a$, $a_{n+4} = \sqrt{2}(b - \sqrt{2}a) - (\sqrt{2}b - a) = -a$, so $a_{n+4} = -a_n$, hence $a_{n+8} = a_n$.
Solution 8. $a(x) = \frac{1}{1-2x}$, $b(x) = \frac{x}{(1-x)^2}$, $c(x) = \frac{2x}{(1-2x)^2}$, $d(x) = \frac{x^2}{(1-x)^3} + \frac{x}{(1-x)^2}$, $e(x) = \frac{1}{1-x^2}$
Solution 9. The generating function is $f(x) = \frac{x}{1-2x+x^2} = \frac{x}{(1-x)^2}$. As in Problem 8 this equals $x + 2x^2 + 3x^3 + \ldots$, so $a_n = n$.
Solution 11. For $n = 1, \ldots, 8$ we have $o_n = d_n = 1, 1, 2, 2, 3, 4, 5, 6$.
Solution 14. You cannot be sure. One possibility is $1, 2, 4, 8, 16, \ldots$, that is, $a_n = 2^n$. But the formula $a_n = (n^2 + n + 2)/2$ also yields $a_0 = 1$, $a_1 = 2$ and $a_2 = 4$. But $a_3 = 7$.
Suggestion for further study (if you know binomial coefficients): Find the values of $\binom{n}{0} + \binom{n}{2} + \binom{n}{4}$ for $n = 1, \ldots, 5$ and then for $n = 6$. Generalization? Reason?
Solution 15. There are various patterns. One way to do it is as follows: notice that in $a_0 = \frac{2}{1}$, $a_1 = \frac{5}{4}$, $a_2 = \frac{41}{40}$ the sum of numerator and denominator is always a power of three: $3, 9, 81$. The exponents are $1, 2, 4$, so 2^n (at least for $n = 0, 1, 2$). In addition, the difference between numerator and denominator is 1. Looking for numbers N, D satisfying $N + D = 3^{2^n}$ and $N - D = 1$ we get $N = \frac{1}{2}(3^{2^n} + 1)$, $D = \frac{1}{2}(3^{2^n} - 1)$. So we guess that $a_n = \frac{3^{2^n}+1}{3^{2^n}-1}$ for all n.
By a short calculation you can check that this satisfies the recurrence relation and initial condition, so it is correct for all n.
Suggestion: The core of the calculation is $\frac{1}{2}\left(\frac{x+1}{x-1} + \frac{x-1}{x+1}\right) = \frac{x^2+1}{x^2-1}$. That is, if $a_n = \frac{x+1}{x-1}$ then $a_{n+1} = \frac{x^2+1}{x^2-1}$. Use this to find a closed formula for any

initial value a_0.

Solution 16. $a_0 = 1$, $a_{n+1} = \frac{1}{2}\left(a_n + \frac{x}{a_n}\right)$.

Solution 17. The fixed point equation is $x = 2\sqrt{x}$, whose only positive solution is $x = 4$. This suggests to compare a_n to 4. Using induction we get $a_n < 4$ for all n. Also $a_{n+1} - a_n = \sqrt{a_n} - \sqrt{a_{n-2}} = (\sqrt{a_n} - \sqrt{a_{n-1}}) + (\sqrt{a_{n-1}} - \sqrt{a_{n-2}})$, so inductively $a_{n+1} \geq a_n$ for all n. Therefore, (a_n) converges to 4.

Solution 18. It seems obvious that $0 < \sqrt{6} < \sqrt{6 + \sqrt{6}} < \dots$, so $a_0 < a_1 < a_2 < \dots$. We need to show that the sequence (a_n) is bounded above. What's a good candidate for an upper bound? Let us try a solution of the fixed point equation. Solving $x = \sqrt{6 + x}$ we get $x = 3$ as unique positive solution.

So we first show $a_n < 3$ for all n by induction. Using this we get a formal proof that $a_{n+1} > a_n$. Therefore the sequence converges to 3, i.e. the infinitely nested root has the value 3.

Solution 19. No. If it was bounded then it would have to have a limit since it is obviously increasing. For the limit we would have $x = x + \frac{1}{x}$ which is impossible.

Other solution: $a_{n+1}^2 = a_n^2 + 2 + \frac{1}{a_n^2} > a_n^2 + 2$, so $a_n^2 \geq 1 + 2n$ for all n. This also shows that the sequence a_n diverges at least like $\sqrt{n}$.

Solution 20. $\frac{1}{\sqrt{5}}\alpha^{12} = F_{12} + \frac{1}{\sqrt{5}}\beta^{12}$ and $F_{12} = 144$, $|\beta| < 0.7 \Rightarrow \beta^2 < 0.49 < \frac{1}{2} \Rightarrow \beta^{12} < \frac{1}{64} \Rightarrow 0 < \frac{1}{\sqrt{5}}\beta^{12} < \frac{1}{100}$, so $\frac{1}{\sqrt{5}}\alpha^{12} = 144.00\dots$.

Solution 21. $a_1 = 1$, $a_2 = 2$, $a_n = a_{n-1} + a_{n-2}$ (distinguish tilings that start with a vertical domino on the left or with two horizontal dominoes). So $a_n = F_n$ for all n.

Solution 22. For any subset $A = \{a_1, \dots, a_k\} \subset \{1, 2, \dots, \lfloor\sqrt{n}\rfloor\}$ where $a_1 < \dots < a_k$ and $k \geq 0$ consider the partition $n = a_1 + \dots + a_k + r$ where $r = n - a_1 - \dots - a_k$. The main point is to note that $a_1 + \dots + a_k \leq 1 + \dots + \lfloor\sqrt{n}\rfloor = \lfloor\sqrt{n}\rfloor(\lfloor\sqrt{n}\rfloor - 1)/2 \leq \sqrt{n}(\sqrt{n} - 1)/2 < n/2$ so that $r > n/2 \geq \lfloor\sqrt{n}\rfloor$ where the last inequality holds for $n \geq 2$. Therefore the partition $n = a_1 + \dots + a_k + r$ is written in increasing order, and this shows that any two different subsets A will give different partitions. Therefore $p_n \geq 2^{\lfloor\sqrt{n}\rfloor}$.

8 Further reading

There are many good books on problem solving. *Arthur Engel*'s book (**Problem-Solving Strategies**) has lots of problems (and hints/solutions) at all levels (also on sequences).[15] I also like *Paul Zeitz*'s book (**The Art and Craft of Problem Solving**).[16] Finally, *Daniel Grieser*'s (yes, that's me) book (**Exploring Mathematics – Problem-Solving and Proof**) introduces many problem-solving techniques (with many explicitly solved problems) and prepares at the same time for university style mathematics.

[15] Arthur Engel trained, very successfully, the German IMO team for many years in the 1970s and 1980s.

[16] Paul Zeitz has trained, very successfully, the US IMO team.