Berkeley Math Circle **De Bruijn, Dominoes, Decks, & Dilworth** **Ted Alper**
September 4, 2019 **tmalper@stanford.edu**

I have two different math circle topics. They aren't *closely* related, though they have some things in common. Either one might be enough for the evening, but I thought one of them might only be enough for half a circle, so I came prepared for both.

1 De Bruijn sequences, chains of dominos, and the bridges of Königsberg

Problem 1 (A card trick). I need five volunteers, in a row, left-to-right. The leftmost volunteer takes the deck and may cut it as much as he/she likes (a cut is taking a stack of cards from the top of the deck and placing them at the bottom). Once satisified, he or she takes the top-most card and then hands the deck to the next volunteer, who takes the top-most remaining cards and hands it to the next volunteer, etc. Once all five volunteers have taken the top card from the deck, my job will be to guess their cards. I may need a little help…

A *de Bruijn sequence of order n* is a sequence of letters (or numbers or other symbols) in which every possible sequence of length n of those letters occurs exactly once (the sequence is cyclic, if you run off the end of the sequence you go back to the beginning). So for example the sequence: 0110 is a de Bruijn sequence of order 2 for the symbols "0" and "1") since the first two digits are 01, the second two are 11, the third two are 10 and the fourth two (wrapping around) are 00.

(1) Can you find a de Bruijn sequence of order 3 for the symbols "0" and "1"?

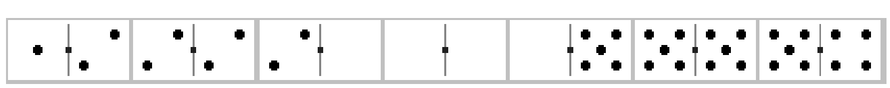
(2) Can you find a de Bruijn sequence of order 2 for the symbols "0" and "1" and "2"?

(3) Can you find a 100 digit number (in base 10) in which every possible 2-digit sequence appears (again, allowing wrapping around – this is just asking for a de Bruijn sequence of order 2 for the digits 0–9)?

(4) In general, can you always find a de Bruijn sequence of any length on any finite set of symbols? If not, describe when you can and when you can't. (you do not need to solve this one to continue with the other questions)

Problem 2 (dominoes) Each 1×2 domino comes with a pair of numbers on it (with blank representing 0), the same pair occurs exactly once in a set of dominoes. "Double Sixes" dominoes use the numbers 0–6, "Double Nines" use the numbers 0–9, "Double Twelves" use the numbers 0–12. They don't sell them, but we could imagine a set of "Double Threes".

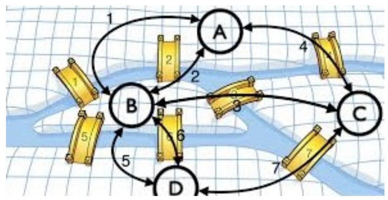
(1) How many dominoes are in a set of Double Sixes? Can you arrange them all in a line, in which adjacent dominoes have the same number on their adjacent sides? Can you do it so the last number on the left side agrees with the last number on the right side?



(2) How many dominoes are in a set of Double Nines? Can you arrange them all in a line, in which adjacent dominoes have the same number on their adjacent sides?

Problem 3 (bridges) Finally, are you familiar with the story of the Bridges of Königsberg? From Wikipedia:

> The city of Königsberg in Prussia (now Kaliningrad, Russia) was set on both sides of the Pregel River, and included two large islands - Kneiphof and Lomse - which were connected to each other, or to the two mainland portions of the city, by seven bridges. The problem was to devise a walk through the city that would cross each of those bridges once and only once.



What might this have to do with the two other problems?

2 Partial Orders, Dilworth's Theorem, Hall's Marriage Theorem

These notes are fragmentary; I haven't written every proof and definition in detail – I'd like to work the details out together with you in class!

Linear ordering may seem like a strange topic – after all, it's just a bunch of objects, placed in a row! But they can be surprisingly subtle, and they crop up in a lot of branches of mathematics – among them, set theory, logic, recursion theory, topology, analysis, graph theory and combinatorics.

Today the *main* goal is to learn enough to appreciate and prove a theorem about partial orders and then see how it can be applied to a number of seemingly-unrelated theorems in graph theory, combinatorics, and linear algebra.

2.1 Warm-up Problem

Problem 4 (a different card "trick") I need 13 volunteers. We deal out a full deck of 52 playing cards to the volunteers, each gets 4 cards. I'd like to have them each give me a card with a different rank (Ace, 2, 3, 4,..., J, Q, K). Is this always possible? Or can we find a way to give them out 4 cards each so that we can't get a different card from each person? Of course, if we *can* do it, they are left with 3 cards each. If so, can I *again* manage to get a different card from each person? And if so, could we do it when they each have only two cards left? (of course, if we get them down to one card each, we can surely finish).

Could we do the same game with four volunteers, thirteen cards each, and I want to get a different *suit* from each of them? (Clubs, Diamonds, Hearts, Spades) – can we do that thirteen times?

2.2 Total Order and order types

The formal definition of an order looks like:

Definition 1 *A* linear ordering of the set *A is a binary relation* $\preceq$ *on A satisfying the conditions:*

transitivity *if* $a \preceq b$ *and* $b \preceq c$*, then* $a \prec c$.

antisymmetry *if* $a \preceq b$ *and* $b \preceq a$*, then* $a = b$.

reflexivity $a \preceq a$.

connectivity *For all a and b, either* $a \preceq b$ *or* $b \preceq a$.

Other terminology: A linear order is sometimes called a *total order*. Also a *chain* (see next section. where we also use the expression *anti-chain*).

We should try to think of some different examples to make clear what this definition means. **Examples:** The integers, the real numbers, the rational numbers, ordered the way we always order them. But we can think of other examples, like all polynomials with real coefficients, or words in a dictionary – wait, can we compare them to real numbers? (Lexicographic order on sequences) What if the words were infinitely long? Are two total orders of the same size *isomorphic*? What does "isomorphic" mean here?

Notice that connectivity implies reflexivity. Why did we bother to include reflexivity as an axiom?

Problem 5 Show any countable linearly-ordered set is embeddable in the rationals, under the usual order.

Problem 6 If $\langle A, \preceq_1 \rangle$ is embeddable in $\langle B, \preceq_2 \rangle$ and $\langle B, \preceq_2 \rangle$ is embeddable in $\langle A, \preceq_1 \rangle$, does that mean that the two must be isomorphic?

2.3 Partial Orders

Problem 7 A group of one hundred students, with no two exactly the same height, were arranged in a square formation. In each of the ten rows, the shortest student raised his or her hand – of these students, John was the tallest. Then, in each of the ten column, the tallest student raised his or her hand; of these, Mary was the shortest. Who is taller, John or Mary?

Problem 8 The same hundred students mentioned in the warmups above, still with no two of exactly the same height, were marching in a single column. Prove that *either* you can find ten students (not necessarily consecutive) in the column whose are in ascending order (that is, the smallest student is in front of the others, the second smallest is next, etc.) *or* you can find *twelve* students (again, not necessarily consecutive) in the column whose heights are in *descending* order (that is, the tallest is in front, etc.).

More generally (and yet also either to look in detail at smaller cases)

Problem 9 (Erdős-Szekeres Theorem) If $mn + 1$ students, all of different heights, are arranged in a straight line from left to right, prove that there must either be subsequence (what does that mean?) of $m+1$ students, whose heights from left to right are increasing; or a subsequence of $n+1$ students, whose heights from left to right are decreasing.

Maybe for an example, it would be easier to do it with 10 students (and subsequences of length four) – too many to just brute-force the idea, but small enough to learn a technique.

Definition 2 *A* partial order *satisfies transitivity, reflexivity, and antisymmetry as in the definition for linear orders above, but need not satisfy connectivity.*

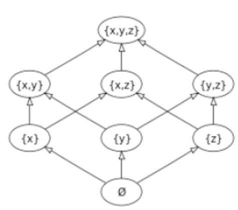
A *chain* is a totally-ordered subset of a partial order. An *anti-chain* is a subset of a partial order in which no two elements are

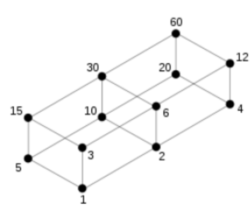
Other terminology:

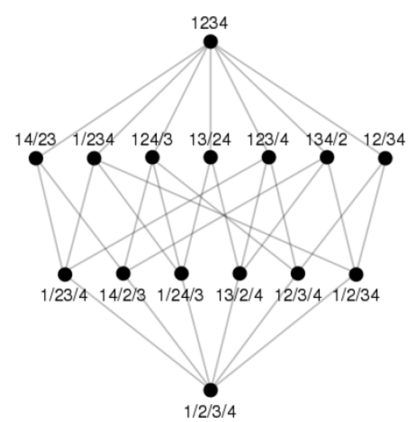
- some people call a partially ordered set a poset.
- a *weak* partial order satisfies transitivity and reflexivity, but not necessarily antisymmetry.
- Elements a and b of the partially ordered set A are said to be *comparable* if either $a \preceq b$ or $b \preceq a$. (so, in a linear order, all pairs of elements are comparable). Elements that are not comparable are called *incomparable*.
- a *chain* is a subset of partially ordered set in which the partial order is linear [that is, all pairs of elements of the subset are comparable].
- An *anti-chain* is a subset of the partially ordered set in which no two distinct elements are comparable (The elements are pairwise incomparable).

Examples. Set inclusion, integers and divisibility, functions and asymptotic dominance, SAT scores, those hundred students in a row, a sequence of tasks to be done in a project, some of which must be done before others, a directed acyclic graph (what's that?) ordered by reachability (what's that?), partitions of a set ordered by coarseness.

We could also draw some (hopefully, self-explanatory) *Hasse diagrams*:







Problem 10 Given an infinite sequence of rational numbers, show there is either a monotonically increasing (infinite) subsequence, a monotonicaly decreasing (infinite) sequence, or a constant (infinite) sequence.

Problem 11 (Questions we won't cover today)

- How many different partial orders on n elements are there? (Sloane's integer sequences A001035, it begins $1, 1, 3, 19, 219, 4231, 130023, \ldots$). Note that for total orders, this is a *much* simpler question.
- Can every partial order be extended to a total order? What does "extended" mean heare?

- Given a partial order, in how many ways can it be extended to a total order?
- (Open question! the 1/3 - 2/3 Conjecture) In any partially ordered set that is not a linear order, there is some pair of elements (x, y) such that the proportion of linear extensions in which $x \prec y$ is between 1/3 and 2/3.

2.4 Dilworth's theorem

Theorem 3 (Dilworth) *If every antichain in a (finite) partially ordered set has at most m elements, then the set may be partitioned into m chains.*

First, there's a much easier theorem that seems similar:

Theorem 4 (Mirsky) *If every chain in a partially order set has at most m elements, then the set may be partitioned into m antichains.*

This can be proved by defining a concept: the *height* of an element. How do you think this should be defined? And how does that lead to a proof of this theorem? (And why doesn't that proof carry over to Dilworth's theorem?)

Some ways to prove Dilworth's theorem: Use induction on the number of elements, or use induction on the number of edges of the associated graph, use induction on the size of the antichain. (Do we need to have a digression on induction?)

Problem 12 (induction example) Show that (for any integer $n > 1$):

$$\frac{1}{1\cdot 2} + \frac{1}{2\cdot 3} + \frac{1}{3\cdot 4} + \ldots + \frac{1}{(n-1)\cdot n} = \frac{n-1}{n}$$

.

Problem 13 (induction example) Show that every number of the form 1007, 10017, 100117, 1001117, etc. is divisible by 53.

2.5 Consequences of Dilworth's theorem

Problem 14 [(All [Soviet] Union Mathematical Olympiad 1972)] Fifty line segments lie on a common line. Show that either some eight of the segments have a non-empty intersection, or eight of the segments are pairwise disjoint.

This well known theorem is a very quick consequence of Dilworth's theorem:

Theorem 5 (Hall's marriage theorem) *Given a collection of people with n men and n women with the property that, for any subset of k men (where k could be any integer between 1 and n), there are at least k women known to at least one man in the subset, then there must be a way to pair each man with a distinct woman known to him.*

Sketch of proof: Define a partial ordering in which each man is "less than" every woman he knows. How big are the biggest antichains?

Go back to our "card trick" puzzle. Can you see how that relates to Hall's marriage theorem?

Problem 15 In a $2n \times 2n$ chessboard, there are n rooks in each row and each column of the board. Show that there exist $2n$ rooks no two of whom are in the same row and same column.

Problem 16 (Putnam 2012) A round-robin tournament of $2n$ teams lasted for $2n-1$ days, as follows. On each day, every team played one game against another team, with one team winning and one team losing in each of the n games. Over the course of the tournament, each team played every other team exactly once. Can one necessarily choose one winning team from each day without choosing any team more than once? (Hint: is this like Hall's marriage theorem? Who is "marrying" whom?)

Dilworth's theorem is closely related to many theorems in graph theory

For example, this is sort of a generalization of Hall's marriage theorem, though it can be proved with Dilworth's theorem too:

Theorem 6 (König's theorem) *In a bipartite graph, the number of edges in a maximum matching equals the number of vertices in a minimum vertex cover.*

What's a graph? a *bipartite* graph? a maximum matching? a minimum vertex cover?

Or another example, involving the maxflow and min cut theorem of directed graphs – which requires some terminology and examples to make clear)

Theorem 7 (Elias-Feinstein,Shannon, also Ford-Fulkerson) *In a flow network with source s and sink t, the maximum achievable flow is equal to the minimum capacity of a cut.*

What's a flow network? a source? a sink? a flow? a cut? a capacity? Is this result trivial? What does it have to do with partial orders? (It helps to only use integers in the capacities, but this is not a serious problem.)

Theorem 8 (Birhoff-von Neumann) *Any doubly-stochastic matrix may be represented as a convex combination of permutation matrices*

Wait! What's a doubly-stochastic matrix? What's a permutation matrix? What's a convex combination? And what on earth does this have to do with partial orders? [actually, I'm not sure that it does have anything to do with them, but I do know a proof that uses the Hall Marriage Theorem. But what on earth does this have to do with men and women? A permutation matrix "marries" each row to a column.]

While we're at it: How big is the biggest anti-chain for the partial order defined by inclusion on the set of all subsets of n elements?

Theorem 9 *(Sperner) if $\mathcal{A}$ is a family of subsets of n elements with the property that no element of the family is contained in any other, then*

$$|\mathcal{A}| \leq \binom{n}{\lfloor n/2 \rfloor}$$

(and this is achieved by letting $\mathcal{A}$ be the collection of all subsets of size $\lfloor n/2 \rfloor$).

Proof, via the LYM (Lubell, Yamamoto, Meshalkin) inequality:

$$\sum_{A \in \mathcal{A}} 1/\binom{n}{|A|} \leq 1$$