BMC Intermediate II, April 2019
Determinants and Path Counting

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Our first goal today is to discuss some matrix basics – determinants and matrix multiplication. We will use these to explore linear transformations in the plane, and then to start building up to my favorite math theorem ever.

- First, let's do the determinant of 2×2 matrix.

- Next, let's see the recursive method for a 3×3 determinant based on 2×2 determinants.

- How about a shortcut for 3×3 determinants?

- What would you guess for a 4×4 determinant? 5×5? 6×6?

- Let's focus on things in the plane for a moment. To let matrices act on points, we need to know how to multiply matrices.

- Let's experiment. Start with some basic shapes in the plane (perhaps squares, rectangles, or triangles), and let's see how a given matrix transforms these shapes. I recommend the use of colored pencils if you have some. I'll suggest some examples on the board. You can do computations here and sketch on the provided graph paper.

- See if you can find 2x2 matrices which perform the following transformations. Find the determinant of each one. Do you notice anything special?
  - Reflect over the x-axis.
  - Reflect over the y-axis.
  - Reflect over the line $y = x$.
  - Reflect over the line $y = -x$.
  - Rotate 90° counterclockwise about the origin.
  - Rotate 180° counterclockwise about the origin.
  - Rotate 270° counterclockwise about the origin.
  - Dilate/scale all lengths by a factor of 3.
  - Dilate/scale all lengths by a factor of 3 and rotate the shape 180°.

- Actually all rotation matrices fit a very specific form, which we'll discuss on the board. Try it out for some different angles.

- There is a nice to find areas of triangles using determinants. Let's explore.

Next we'll be doing a little algebraic combinatorics and graph theory to set us up for one of my favorite theorems (and if time allows, some fancy variations).

Problems:

1. Find the path matrix of each of the graphs below and compute its determinant. (These ones should be 2×2, with two sources on the left and two sinks on the right.)

2. Find the path matrix of each of the graphs below and compute its determinant. (These ones should be 3×3, with three sources on the left and three sinks on the right.)

3. Examine the examples you computed in Problems 1 and 2. You may also want to try a few examples of your own or some of the extra graphs on the back of page (no need to turn in work for extras). Do you have any conjectures about the link between paths in the graph and the determinant of the path matrix? Briefly explain any patterns you see. (Feel free to discuss with classmates, but don't do any internet research yet. Tomorrow we will see the theorem and its proof, but it's more fun if you can discover at least the statement on your own.)

4. Bonus: Construct a directed graph with 2 sources and 2 sinks whose path matrix is
$$\begin{pmatrix} a & b \\ c & 0 \end{pmatrix},$$
where a, b, and c are nonzero. As a challenge to yourself, try to use as few edges as possible.