

## STRANGE DISTANCE FUNCTIONS AND STRANGER CIRCLES, PART II

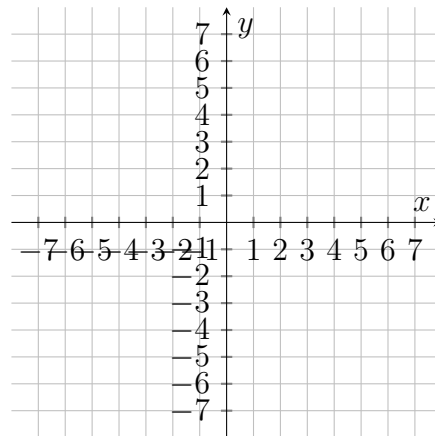
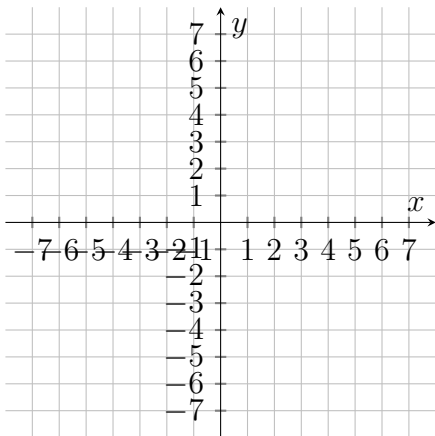
Last week we studied taxicab distance on a square grid of roads. Today we will explore some more distance functions in the plane. We will ask the same kinds of questions as last time.

### 1 Hogwarts Owl Letter Delivery (OLD) Metric

Suppose that magical owls delivering letters must always stop off at the Owl Post Office in Hogsmeade. Thus, if Harry wants to send an owl to Ron, his owl must first fly to the Owl Post Office, and then fly to Ron. (It cannot fly straight from Harry to Ron without checking in at the Post Office.) As owls are smart and can fly, they will take straight line paths for each segment they fly. Also, the owls will refuse to let you mail a letter to yourself; instead they insist you just hand it to yourself. (Consider all points in the plane this time, not just the grid of roads, and assume the Owl Post Office is at the origin and each block is 1km by 1km.)

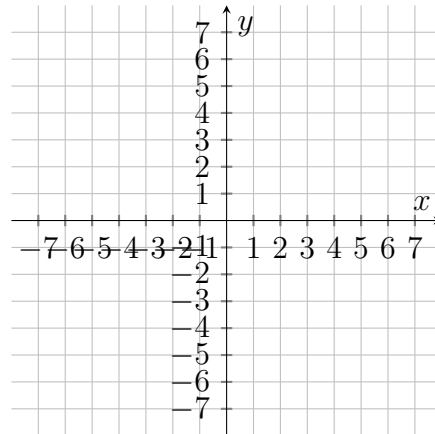
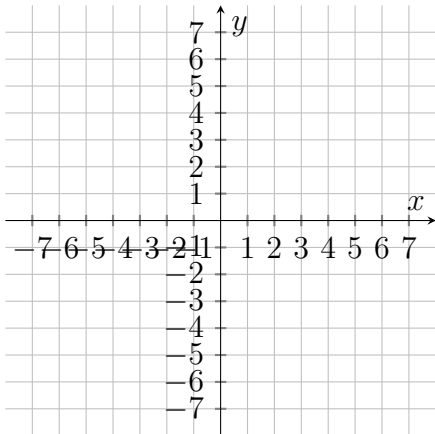
Any questions?

- On the first grid below, plot Hermione at  $(4, 3)$  and Dobby at  $(-4, 4)$ . On the second grid below, plot Hermione at  $(4, 3)$ , Ron at  $(5, 3)$ , Snape at  $(0, -6)$ , Malfoy at  $(5, 0)$  and Dobby at  $(-4, 4)$ .

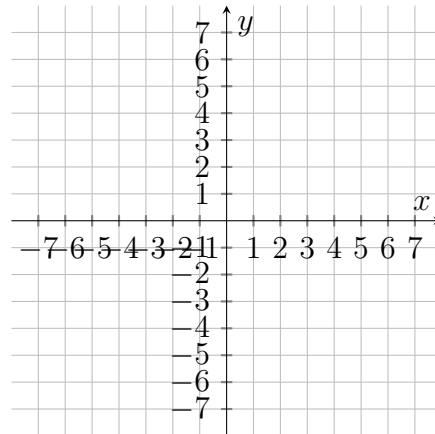
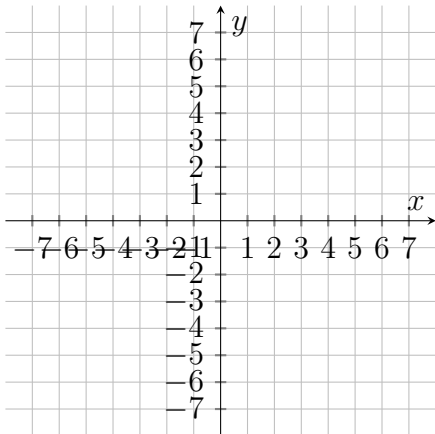


- Suppose Hermione needs to send an urgent letter by owl to Dobby. Is there only one shortest path her owl can take, or are there several? How far must her owl fly on the journey? This will be the OLD distance.
- Which character above is closest to Hermione by the OLD distance (i.e. the shortest distance the owl can fly, including the stop at the post office)? Make a guess just by looking at the figure, and then check the actual computations to justify or fix your answer.

4. Let's think about the triangle inequality now. Can you find three points  $A$ ,  $B$  and  $C$  so that the OLD distance from  $A$  to  $B$  plus the OLD distance from  $B$  to  $C$  EQUALS the OLD distance from  $A$  to  $C$ ? Show an example on the left grid below. (You can denote this  $AB_{OLD} + BC_{OLD} = AC_{OLD}$  if you like.) (*Bonus HW: describe ALL such pairs of points, and explain why you have them all.*)

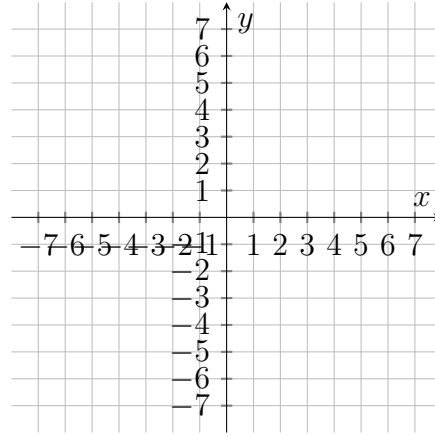
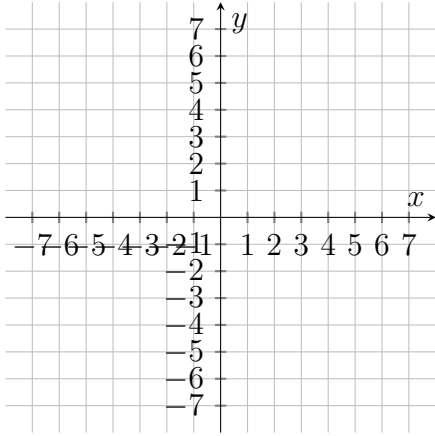


5. Time for circles. Let's start with OLD circles around the origin, AKA Owl Post Office. Sketch the points which are OLD distance 1 from the origin, or in other words, points the owl can reach by flying a distance of 1km. Repeat for 2km, 3km, etc.
6. That seemed too good to be true. Let's focus on Hermione again, on the left grid below. Can you find any points which are OLD distance 1 from Hermione? OLD distance 2? Maybe 3, 4, 5, 6?

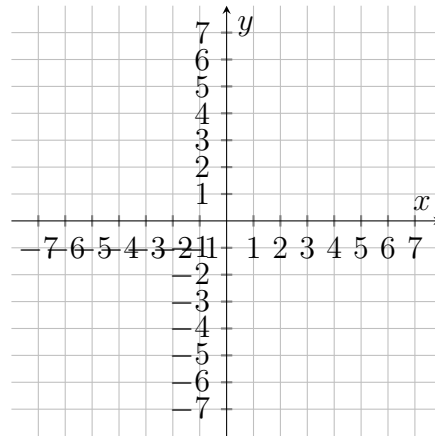
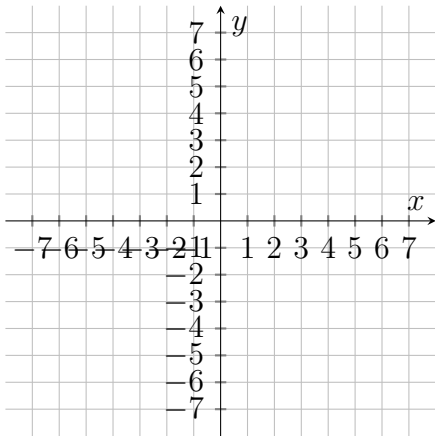


7. From our previous work, can we sketch some OLD circles centered at Hermione? What is weird about these circles?

8. Recall that Hermione is at  $(4, 3)$ , Malfoy is at  $(5, 0)$ , and Dobby is at  $(-4, 4)$ . Are there any points which are OLD equidistant from Hermione and Malfoy? Are there any points which are OLD equidistant from Malfoy and Dobby? Use the grids below to experiment.



9. If  $A, B, C,$  and  $D$  are four points such that  $AB$  and  $CD$  are equal Euclidean distances, must it also be true that  $AB_{OLD} = CD_{OLD}$ ? If yes, explain why. If no, show an example below on the left.



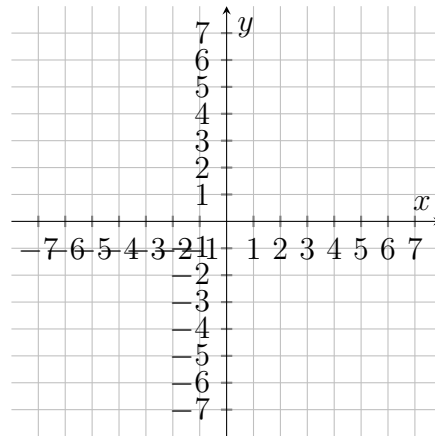
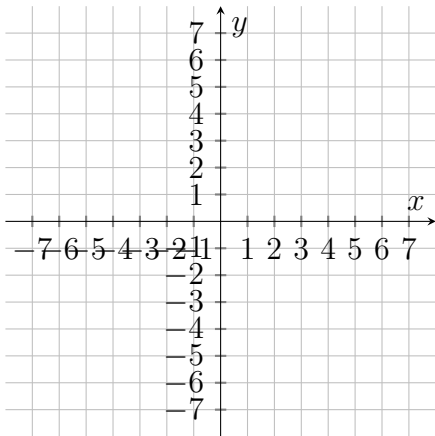
10. If  $A, B, C,$  and  $D$  are four points such that  $AB_{OLD}$  and  $CD_{OLD}$  are equal Owl Letter Delivery distances, must it also be true that  $AB = CD$  in Euclidean distance? If yes, explain why. If no, show an example above on the right.

## 2 Teleportation metric

Imagine we have invented a very excellent portable teleporter, and we can teleport from any starting point to any ending point in one move. There is one exception – you cannot teleport from any point to the same point; that is a waste of a precious teleporter. Thus, the distance from a point to itself is 0 (no moves), and the distance between any two *different* points is 1 move.

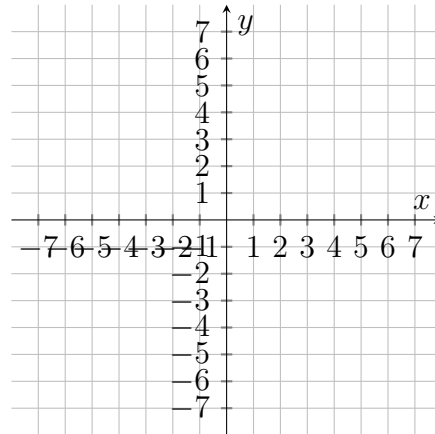
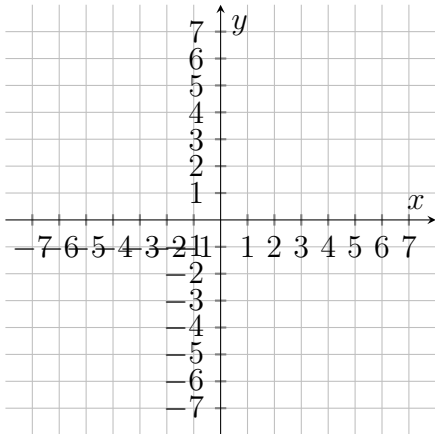
Any questions about the rules?

11. Choose a points  $K$  (other than the origin) on the graph below. If  $P$  is any point, let  $KP_*$  denote the teleporter distance from  $K$  to  $P$ . Which points are on the teleporter circle of radius 1 around  $K$ ? (In other words, which points are 1 teleporter move away from  $K$ ?) Graph this set, and also describe it in words.

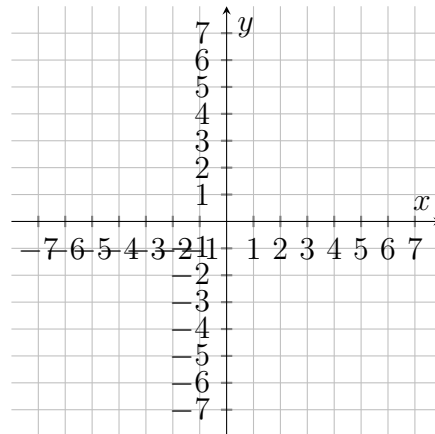
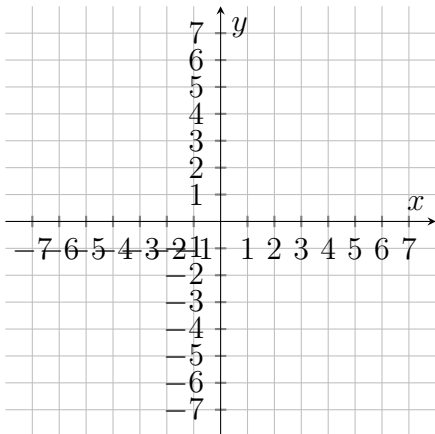


12. Which points are on the teleporter circle of radius 2 around  $K$ ? Radius 3? Radius 4? You can use the extra graph above if necessary.
13. Does the teleporter distance satisfy the Triangle Inequality? I.e. is  $AB_* + BC_* \geq AC_*$ , no matter what points  $A$ ,  $B$ , and  $C$  you use? When can you get equality?

14. Now, some rascal has tinkered with our teleporter, and it can no longer teleport us ridiculously long distances. It can only teleport us a distance of 5 blocks or less in one move. Let  $KP_{\infty}$  denote the weak teleporter distance between  $K$  and  $P$ . Now try the circles again. Find the weak teleporter circle of radius 1 centered at  $K$  on the first graph below, and the weak teleporter circle of radius 2 centered at  $K$  on the next graph below. For radius 2, these are points we can reach with two teleporter moves, but not with only one move. (You may change your point  $K$  if your first choice is inconvenient now.)



15. Does the Triangle Inequality seem to work for the weak teleporter? I.e. is  $AB_{\infty} + BC_{\infty} \geq AC_{\infty}$ , no matter which points we choose? Below are grids for experimenting.



Below are some additional metrics we can study. Come up with your own questions to ask about them. If there is time, you can start working on them now. If you have a really good one, let me know so that perhaps I can share it with the other students next time.

### 3 Chess King metric

Here we imagine an infinite chessboard, represented by an integer grid of points (possible positions for the king). The distance between two points  $A$  and  $B$  is the number of moves it would take a chess king to move from  $A$  to  $B$ .

Any questions about the rules?

Questions to explore (totally fine to list more than 3):

- 16.
- 17.
- 18.

### 4 Infinitely Long and Tall Hotel with Only One Elevator metric

In the Infinitely Long and Tall Hotel with Only One Elevator, there are infinitely many floors (floor 0, floor 1, floor 2, floor 3, etc.), just one infinitely tall elevator at one end of the hotel, and infinitely many rooms on each floor (room 1 closest to the elevator, then room 2, then room 3, and so on). The elevator is the only way to move between different floors – no stairs (eep, fire safety!), no climbing around like monkeys, no drilling holes in the building, and so on. Only the elevator! It takes one move to go up or down one floor on the elevator, and it also takes one move to walk on a given floor from one room to one that is right next to it. Room 1 on each floor is 1 move from the elevator.

Any questions about the rules?

Questions to explore (totally fine to list more than 3):

- 19.
- 20.
- 21.