Berkeley Math Circle Intermediate I, 1/23, 1/20, 2/6
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Topic: Polygons, Polyhedra, Polytope Series

Part 1 – Polygon Angle Formula

Let's start simple. How do we find the sum of the interior angles of a convex polygon?

- Write down as many methods as you can.
- Think about: how can you adapt your method to concave polygons?

Formula for the interior angles of a polygon: _________________ .

Q: There is an equality between the number of ways to triangulate an n+2-gon and the number of expressions containing n pairs of parentheses that are correctly matched. Prove that these numbers are the same. Can you find a closed formula?

Part 2 – Platonic Solids and Angular Defect

Definition: A platonic solid is a polyhedron which a. has only one kind of regular polygon for a face and b. has the same number of faces at each vertex.

One simple example is that of a cube. Each face is a square (=regular quadrilateral) and each vertex is connected to exactly three squares.

In Schläfli notation, this is {4, 3}: the regular polyhedron with 3 regular 4-gons at each vertex.

- Can there be polyhedra with exactly one or two squares at each vertex?
- Can there be polyhedra with exactly four squares at each vertex?

For future reference, we'll need the notion of the "angular defect" as well. Rather than focusing on what is *present* at the vertex, we focus on what is *absent*: the deviation from 360 degrees at the vertex, or "leftover" angle. In this case, it is $360-90\times3=90$.

Q (abstract algebra): Can you find the symmetry group of the cube? Other platonic solids?

Let's prove that there are exactly 5 Platonic solids and no more.

- Be careful – we'll need some way to handle discussing regular n-gons as n trends to infinity.

Platonic solids in Schläfli notation:

Now, we have only described what these figures look like at the vertex. Moving forward, it will be helpful to know how many total vertices, edges, and faces each of these polyhedra has.

Let's talk about the situation of the cube, whose properties we already know very well, and then extend to think about the rest of the solids. The number of faces for each will be given.

Cube:

Solid	**Schläfli #**	**Defect at vertex**	**V**	**E**	**F**
Tetrahedron					4
Cube					6
Octahedron					8
Dodecahedron					12
Icosahedron					20

Q: How can you find the combinatorial properties of a regular polytope from the Schläfli number alone?

Part 3 – Duals, Truncations, Stellations, χ_E

Let's try placing a point in the middle of each face and connecting the points. Maybe something interesting will happen.

What did you find?

Q: What happens to the Schläfli number of a polyhedron when you take the dual?

Two common operations on polyhedra, in addition to taking the dual, are stellation (=replacing each face with a pyramid) and truncation (=replacing each vertex with an appropriate regular polygon). Let's experiment with the snub and stellated cubes:

- Same as before: let's find the vertices, edges, faces, and angular defect at each vertex.

Try something more complicated: the truncated icosahedron, also known as a “soccer ball.”

- Control question – which polygon replaces each vertex?

We’ve produced some data. Let’s put it in a convenient table. Do you know the Euler Characteristic? While we’re at it, we might as well throw that in too.

Solid	**Defect at vertex**	**V**	**Total Defect at all V**	X_E **= V – E + F**
Tetrahedron				
Cube				
Octahedron				
Dodecahedron				
Icosahedron				
Snub Cube				
Stellated Cube				
Soccer Ball				

Scratch work:

The relationship seen above:

Part 4 – Euler Characteristic and Gauss-Bonnet Theorem

We may have a nice pattern over the figures that we’ve put in the table above, but it has only 8 entries! It would be great to have something a little bit more convincing.

Let’s try to find the total angular defect and Euler Characteristic of, say, a donut shape built from cubes.

- Is this the same as finding the same values for a torus? Why or why not?

Total Angular Defect:

Euler Characteristic:

Our hypothesis:

Proof:

- Hint: Parts 1 and 2 will be useful here.

Q (topology): Can you find the Euler Characteristic of a Möbius strip? A Klein bottle?

The angular defect is not merely a nice tool for finding the certain features of polyhedra: it describes curvature and helps to explain how map distortions are created.

Topologically speaking, we will find that all polyhedra with equal euler characteristic are equivalent. This gives us an easier way to calculate the Euler Characteristic of snub and stellated figures for sure!

Part 5 – Digression - Planar Graphs + Knots

When we know that the Euler Characteristic of a solid is 2, that is the precondition for drawing it as a "planar graph."

A planar graph consists of vertices and edges. Each vertex is connected to at least one edge, and each edge connects exactly two vertices. No edges cross. Let's try it with our friend, the cube.

- Remember, your "before" and "after" pictures should have the same number of V, E, and F.

Try it with the snub cube and icosahedron:

We can now ask graph-theoretical questions of our polyhedra. However, it turns out that we can also make these graphs into knots. We'll learn an algorithm through the example of the trefoil knot:

Knot theory is also intimately related to topology. Interestingly, there are knots that cannot be untied in 3-D, but can be "turned over" in the 4th dimension and unknotted. This leads to our next topic.

Part 6 – Polytope Series

We played in 2-D for a while, then 3-D. What comes next? Well… everything.

Let’s take a moment to think about our own questions:

It would be nice to be able to think about things in the wild land of … 4-D! Let’s keep going with our friend the cube. What should a cube be like in n dimensions?

- If we use what we know about making squares from line segments and cubes from squares, the question is a little easier.

Let’s use some combinatorial tools to learn about the n-dimensional cube analogue’s combinatorial properties. Draw a table below:

Space here for combinatorial proof:

- We’re lucky that Cartesian coordinates are set up with right angles in mind, huh?

Incidentally, the Schläfli number for the 4th dimensional cube (tesseract) is {4, 3, 3}.
Q: Make sure you know how to extend this to the n-dimensional analogue.
Q: How could you extend the Gauss-Bonnet Theorem to higher dimensions?

Let’s try doing a similar process with the tetrahedron.

- How do we make a triangle from a line segment?

Fact: There are no regular polyhedra in 5 dimensions and higher other than the hypercube {4, 3, 3, …, 3}, simplex {3, 3, 3, …, 3}, and cross-polytope {3, 3, 3, …, 4}.

Q: We proved that there were 5 regular polyhedra in 3 dimensions, and the fact above discusses the 5-dimensional case. What about 4 dimensions?

Further reading:

- Alexander Givental, “Geometry of Surfaces and the Gauss-Bonnet Theorem:” https://math.berkeley.edu/~giventh/difgem.pdf
- Coxeter, *Regular Polytopes*.
- Ueno, Shiga, Morita, *A Mathematical Gift, I: The interplay between topology, functions, geometry, and algebra*