Series

BMC Adv Fall 2019

November 20, 2019

1 Warm-Up

Exercise 1.1. 1. What is $\displaystyle\sum_{n=1}^{\infty} \frac{1}{2^n}$?

2. What about $\displaystyle\sum_{n=1}^{\infty} \frac{1}{3^n}$?

3. Can you find a pattern for $\displaystyle\sum_{n=1}^{\infty} \frac{1}{r^n}$ for some r?

4. Does this formula work for all r?

Exercise 1.2. Find $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n}$.

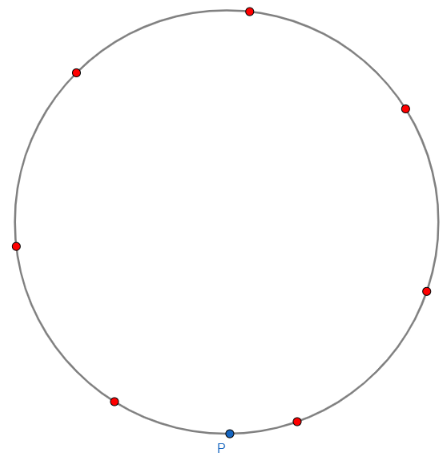
Exercise 1.3. Find $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n(n+1)}$.

Exercise 1.4. Using the previous exercise, prove that $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^2}$ **converges**, i.e. its sum is a finite number and not infinity.

2 An Exact Solution

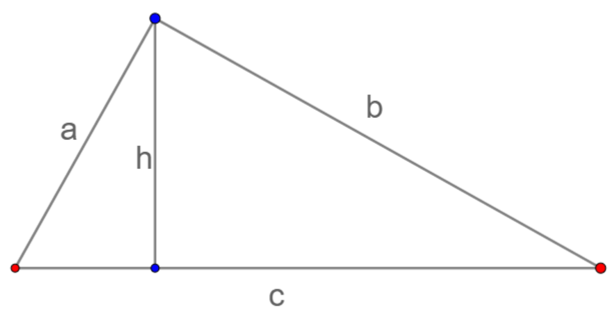
Definition 2.1. A physics fact is that the light you receive at a point falls off as 1 over the distance to the light source squared. So, if we were half as far away from the sun, we would actually get 4 times the light. Let $f_N(x)$ denote how much light you receive if there are N evenly spaced identical light sources on a circle of circumference N, and you are x away from the closest one along the circumference, where $0 < x \leq \frac{1}{2}$.

In the example picture, x is the distance of P from its closest red point and the amount of light received at P is $f_7(x)$. The circle is of circumference 7.

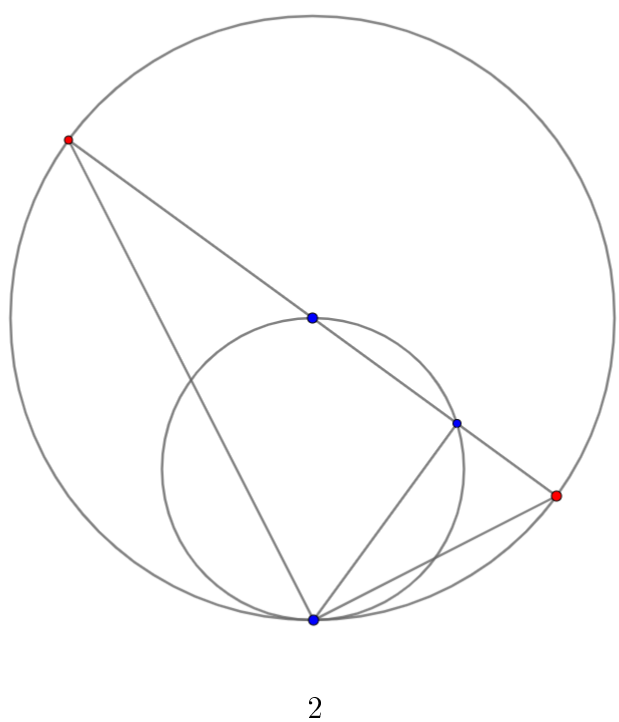


Exercise 2.2. The light received by a source of light d away is $\frac{1}{d^2}$. Prove that $f_1(x) = \dfrac{\pi^2}{\sin(\pi x)^2}$.

Exercise 2.3. In the following picture, prove that $\dfrac{1}{a^2} + \dfrac{1}{b^2} = \dfrac{1}{h^2}$.



Exercise 2.4. Prove that $f_1(x) = f_2(x)$. As a hint, look at the following picture.

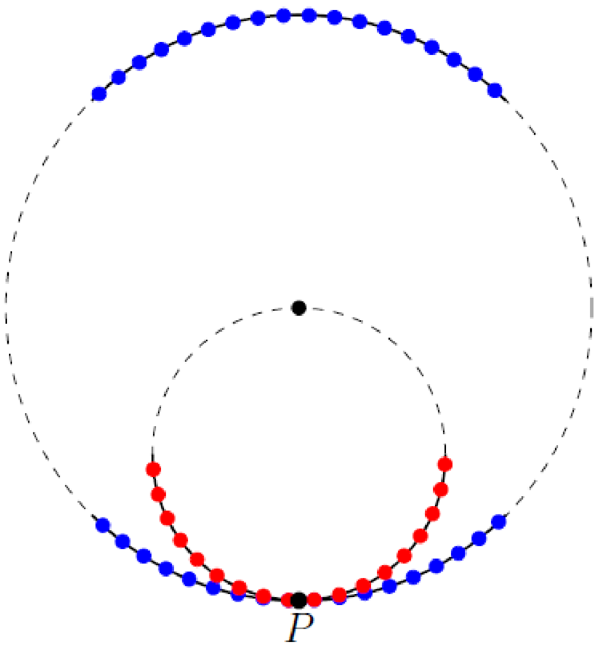


Exercise 2.5. Using the same logic as the previous problem, prove that $f_N(x) = f_{2N}(x)$ for all $N \geq 1$.

Exercise 2.6. Put it all together to prove that $f_{2^N}(x) = \dfrac{\pi^2}{\sin(\pi x)^2}$ for all $N \geq 0$.

3 An Approximate Solution

Definition 3.1. For $k \leq N$, let $f_{k,2N}(x)$ denote the amount of light a point P receives from the nearest k points. A picture is shown below.



Exercise 3.2. Prove that

$$f_{2N}(x) - \frac{\pi^2}{N} \leq f_{N,2N}(x) \leq f_{2N}(x)$$

Exercise 3.3. We want to compare $f_{k,2N}(x)$ and $f_{k,4N}(x)$ and we do so in two steps. Prove that the amount of light P receives from all the red points on the inner circle is equal to the amount of light P receives from the blue points on the outer circle.

Exercise 3.4. The amount of light P receives from the red points is $f_{k,2N}(x)$. The amount of light P receives from the blue points on the bottom half is $f_{k,4N}(x)$. Prove that

$$f_{k,2N}(x) - \frac{k\pi^2}{4N^2} \leq f_{k,4N}(x) \leq f_{k,2N(x)}.$$

Exercise 3.5. Prove that for any N and $j \geq 2$ we have

$$f_{k,2N}(x) - \frac{k \cdot \pi^2}{N^2} \cdot \frac{4}{3} \leq f_{k,2^jN}(x) \leq f_{k,2N}(x)$$

Exercise 3.6. Argue that as you take j to infinity for fixed k that

$$\lim_{j\to\infty} f_{k,2^jN} = \sum_{n=-k/2}^{k/2} \frac{1}{(n-x)^2},$$

and conclude that for any N that

$$f_{N,2N}(x) - \frac{\pi^2}{N}\cdot\frac{4}{3} \le \sum_{n=-N/2}^{N/2} \frac{1}{(n-x)^2} \le f_{N,2N}(x).$$

Exercise 3.7. Put this together with Exercise 3.2 to show that

$$\lim_{N\to\infty} f_{2^N}(x) = \sum_{n=-\infty}^{\infty} \frac{1}{(n-x)^2}.$$

4 Putting it all Together

Exercise 4.1. Use the approximate solution and exact solution to prove that

$$\sum_{n=-\infty}^{\infty} \frac{1}{(n-x)^2} = \frac{\pi^2}{\sin(\pi x)^2}.$$

Exercise 4.2. Plug in $x = \frac{1}{2}$ and prove that

$$\sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{6}.$$

Remark. This is a special value of a more general function called the **Riemann Zeta Function**. It is defined as

$$\zeta(s) := \sum_{n=1}^{\infty} \frac{1}{n^s}.$$

We proved at the beginning that $\zeta(1) = \infty$ and we have just proved is that $\zeta(2) = \dfrac{\pi^2}{6}$.

Mathematicians have found a way to extend the ζ function so you can plug in any complex number.

Conjecture 4.3 (Riemann Hypothesis)**.** If $\zeta(x+iy) = 0$, then $y = \frac{1}{2}$.

Remark. If you have heard of the statement that

$$1 + 2 + \cdots + n + \cdots = \frac{-1}{12},$$

this is because $\zeta(-1) = \frac{-1}{12}$.