Mathematical Cut-and-Paste: An Introduction to the Topology of Surfaces

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A mathematician named Klein
Thought the Möbius band was divine.
Said he, "If you glue
The edges of two,
You'll get a weird bottle like mine."
-Anonymous

Figure 1: A donut Earth. Why not?

Let us begin with a simple question: What shape is the earth? Round, you say? Ok, but round like what? Like a pancake? Round like a donut? Like a soft pretzel? Some other tasty, carbohydrate-laden treat? No, no—it's round like a soccer ball. But how do you know? ...Really, how do you know? Perhaps you feel sure because you've seen photos of the earth from space. Well, people figured out that the earth is round long before we figured out how to build rocket ships (or cameras, for that matter!).

Scientists as far back as the ancient Greeks theorized that the earth is round. Although they offered no substantive proof of their theories, Pythagoras, Plato, and Aristotle were all supporters of the spherical earth theory, mostly based on the curved horizon one sees at sea. Surely this suggests that the earth is not flat like a pancake, but how can we know that the earth isn't some other round shape, like a donut, for example?

If we were to walk around the entire earth, then we can come up with plenty of reasons that it's not shaped like a donut. The most obvious, perhaps, is that if the earth were a donut, there would be some places where we could stand and look directly up into the sky and see more of the earth! Also, there would be places where the curve of the horizon would be upwards instead of downwards. But how can we really, truly know that the shape is that of a ball and not some other strange shape that we haven't yet thought up? As a thought experiment, pretend for a moment that you are locked in a room with thousands and thousands of maps of various places on Earth. Suppose you have enough maps so that you have several for every point on the globe. Could you determine the shape of the earth? Yes! You need only paste together the maps along their overlaps.

This basic idea is exactly the idea that underlies the way mathematicians think about surfaces. Roughly speaking, a *surface* is a space in which every point has a neighborhood that "looks like" a two-dimensional disk (i.e. the interior of a circle, say $\{x, y \in \mathbb{R}^2 \mid x^2 + y^2 < 1\}$.) A sphere is an example of a surface, as is the surface of a donut, which mathematicians call a torus. Some of our everyday, natural notions of surfaces don't quite fit this definition since they have edges, or places where you could fall off if you weren't careful! Mathematically, these are *surfaces with boundary*: spaces in which every point has a neighborhood that looks like either a two-dimensional disk or half of a two dimensional disk (i.e. $\{x, y \in \mathbb{R}^2 \mid x^2 + y^2 < 1, y \geq 0\}$). The old circular model of the earth where you can sail off the edge is an example of a surface with boundary. Another example is a cylinder without a top and bottom.

Figure 2: A surface with boundary

Now that we know what a surface is, let's to try to figure out what kinds of surfaces are out there. Here, we're going to examine this question from a *topological* point of view—we'll be interested in the general *shape* of the surface, not in its *size*. Although the geometric notions of size and distance are quite important in reality, topologists seek to understand the coarser structure of surfaces as a first approximation to understanding their shape. For example, from a topological point of view, a sphere is a sphere, it doesn't matter how large or small the radius is. To this end, we will allow ourselves to deform and manipulate surfaces as if they were made of rubber sheets: we'll consider two surfaces to be the same if we can stretch, shrink, twist, push, or wriggle one surface around until it looks like the other surface. But we will have to be nice in our deformations: topologists aren't so violent as to create holes or break or tear any part of our surface. So, an apple would be considered the same as a pear, doesn't matter if it has a big lump on one end. A flat circular disc is the same as the upper half of the surface of a sphere, even though the latter is stretched and curvy. The classic joke in this vein is that a topologist can't tell the difference between a coffee cup and a donut. If we had a flexible enough donut, we could make a dent in it and enlarge that dent to be the container of the coffee cup, while smooshing (certainly a technical topological term) the rest of the donut down in to the handle of the coffee cup.

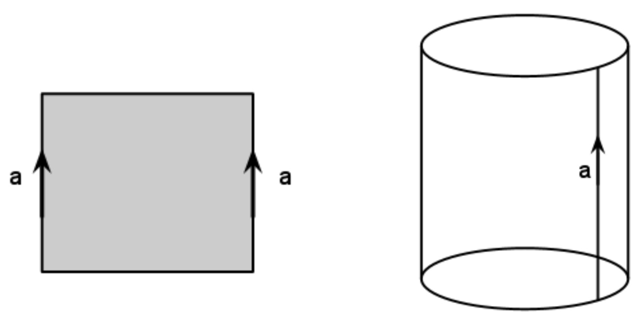
Let's begin by trying to make a list of surfaces that we know. What surfaces can you think of? The first one that comes to mind is the surface of the earth: it's a sphere. (Note here that we're only talking about the surface of the earth, not all the dirt, water, oil, and molten rock that make up its insides! Just the surface—like a balloon.) Another surface that comes up a lot is the *torus*, which is shaped like an innertube. For the most part here, we're going to restrict our investigation to compact (which means "small" in the loose sense that they can be made up of finitely many disks patched together) and connected (made of one piece, i.e. you can walk from one point to every other point on the surface without jumping). We will see some examples of surfaces with boundary because they are surfaces that you may be familiar with. As mentioned before, a cylinder without a top or bottom is a surface with boundary. A Möbius strip is a surface with boundary.

Figure 3: Map of the world [**?**]

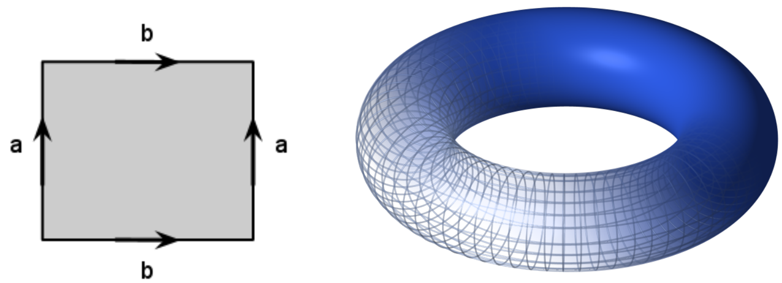
Drawing surfaces on paper or on the blackboard is difficult. One needs quite an artistic hand to convey the shape of an object that lives in our three-dimensional world accurately on two-dimensional paper. However, we'll see that it's easy and quite convenient to record cut-and-paste instructions for assembling surfaces with a simple diagram on a flat piece of paper.

We take our inspiration from maps of the world. In a typical world map, the globe is split open and stretched a bit so it can be drawn flat. We all understand that if we walk out the right side of the map, we come in through the left side at the same height. This is a pretty useful idea! We can imagine a seam on a globe that represents this edge. We can think of taking the map and gluing up the left and right edge to return to our picture of the globe.

There is one slight dishonesty in the typical world map: the representations of very northernly and southernly parts of the earth aren't very accurate. They're much bigger on the map than they are in reality! In fact, the entire line at the top edge of the map really represents just a single point on the globe, the north pole. Similarly for the bottom edge and the south pole. We can make a more honest map by shrinking these edges down so that we have one point at the top and one point at the bottom, representing the north and south poles, respectively. Then our resulting picture is a circle! It has the same properties with respect to walking out through the right edge and coming back in through the left. We can record this information by drawing arrows on the boundary of the circle to indicate how we are to glue up the picture to create a globe. It's a lovely picture: if we glue up one semicircular edge of a circle to the other semicircular edge (without twisting!) then the resulting surface is a sphere. Let's look at some more examples of how this works.

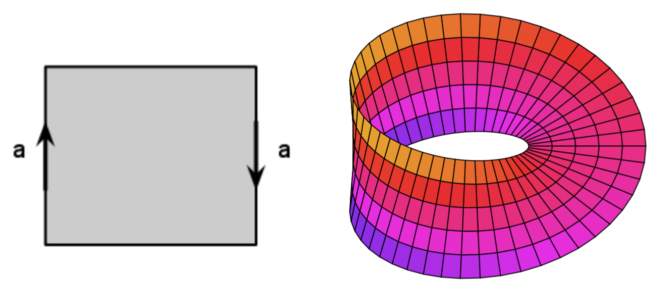


Example 1 (The cylinder)**.** We can create a cylinder by using a piece of paper and gluing the ends together. Thus we can write down instructions for making a cylinder by drawing a square and labeling a pair of opposite edges with a little arrow that indicates gluing them together.



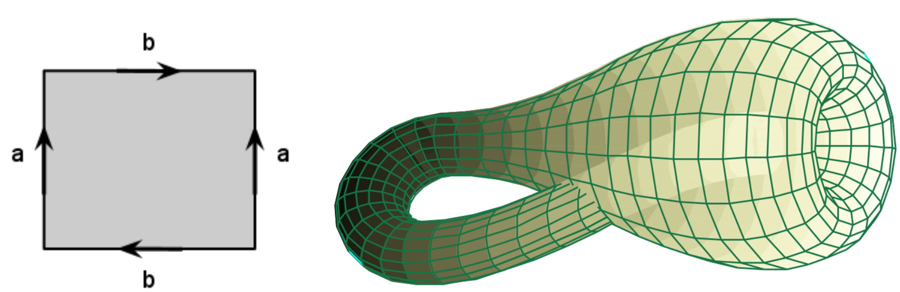
Example 2 (The torus)**.** The diagram above represents a gluing diagram for the torus. To see this, first imagine bringing two of the edges together to form a cylinder. Since the circles at the top and bottom of the cylinder are to be glued together, we can imagine stretching the cylinder around and gluing them to obtain a surface that looks like the surface of a donut. Now, let's practice thinking about how walking around on the surface is represented on the diagram. If we walk out the left edge, we come back in the right edge at the same height. Similarly, if we walk out the top, we come in the bottom at the same left-right position. It's like PacMan!

Exercise 1. Imagine you are a little two-dimensional bug living inside the square diagram for the torus above. You decide to go for a walk. Trace your path. Be sure to exit some of the sides of the square and be careful about where you come back in! Do this several times. Draw some torus gluing diagrams of your own and practice some more.



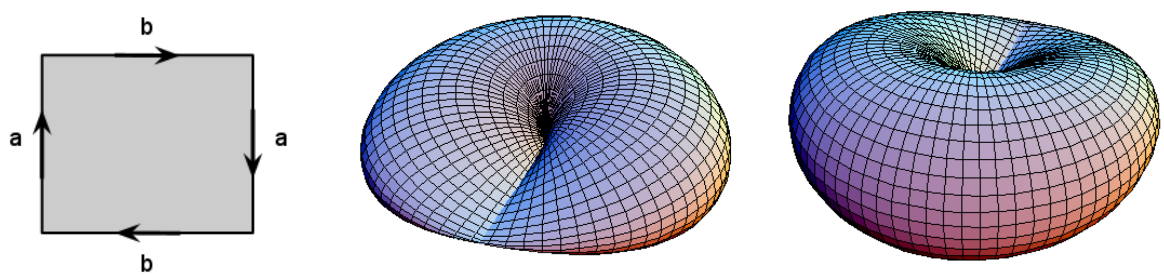
Example 3 (The Möbius Strip)**.** What happens if we start with a square and identify a pair of opposite edges, but this time in opposite directions? The resulting surface is a Möbius strip!

Exercise 2. A cylinder has two boundary circles. How many boundary circles does a Möbius strip have?



Example 4 (Klein bottle)**.** What happens if we reverse the direction that we glue one of the pairs of edges in the diagram that we had for the torus? We can begin by again gluing up the edges that match up to create a cylinder. But now if we try to stretch it out and glue the boundary circles together, we see that the arrows don't match up like they did for the torus! We can't just glue the circles together because our gluing rule says that the arrows must match up. The only way to imagine this is to imagine pulling one end of the cylinder through the surface of the cylinder and matching up with our circle from the inside. The resulting representation of the surface doesn't look like a surface, but it really is! It's funny appearance is just a consequence of the way we had to realize it in our three-dimensional world.

Exercise 3. Imagine you are a little two-dimensional bug living inside the square diagram for the Klein bottle above. You decide to go for a walk. Trace your path. Be sure to exit some of the sides of the square and be careful about where you come back in! Do this several times. Draw some Klein bottle gluing diagrams of your own and practice some more!



Example 5 (The projective plane)**.** What happens if we reverse not just one of the pairs, but *both* of the pairs of edges in our diagram for the torus? The resulting surface is called the *projective plane* and it is denoted $\mathbb{R}P^2$. It's hard to imagine what this surface looks like, but our square diagram will allow us to work with it easily!

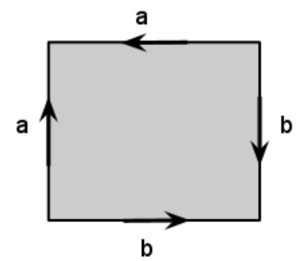
Exercise 4. Imagine you are a little two-dimensional bug living inside the square diagram for $\mathbb{R}P^2$ above. You decide to go for a walk. Trace your path. Be sure to exit some of the sides of the square and be careful about where you come back in! Do this several times. Draw some $\mathbb{R}P^2$ gluing diagrams of your own and practice some more!

Definition 1. A *gluing diagram* for a polygon is an assignment of a letter and an arrow to each edge of the polygon.

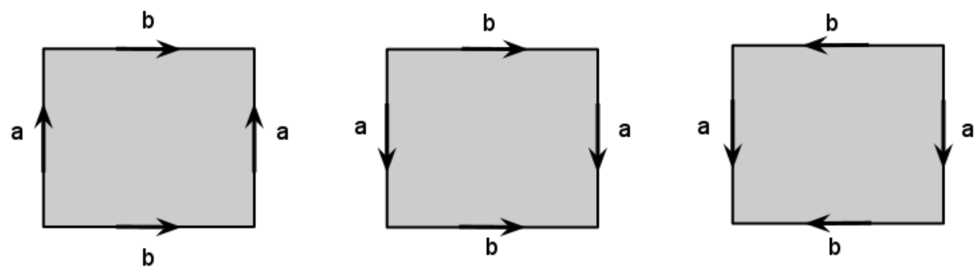
With this general definition, not every gluing diagram represents a surface. For example, if three edges are labeled with the same letter, then these glue up to give something whose cross section looks like ⅄! However, if we assume that the edges are always glued in pairs, then the resulting pasted up object will always be a surface. Here's why. It's clear that every point in the interior of the polygon has a neighborhood that looks like a disk. A point on one of the edges but not on a corner has a neighborhood that looks like a disk if we think about the corresponding point on the edge that it's glued to and draw half-disks around each of them. A point on one of the corners can similarly be given a neighborhood that looks like a disk.

Example 6. The squares that we thought about above for the cylinder, the torus, the Klein bottle, the Möbius strip, and $\mathbb{R}P^2$ are gluing diagrams for these surfaces.

Exercise 5. What surface is represented by the gluing diagram at right?

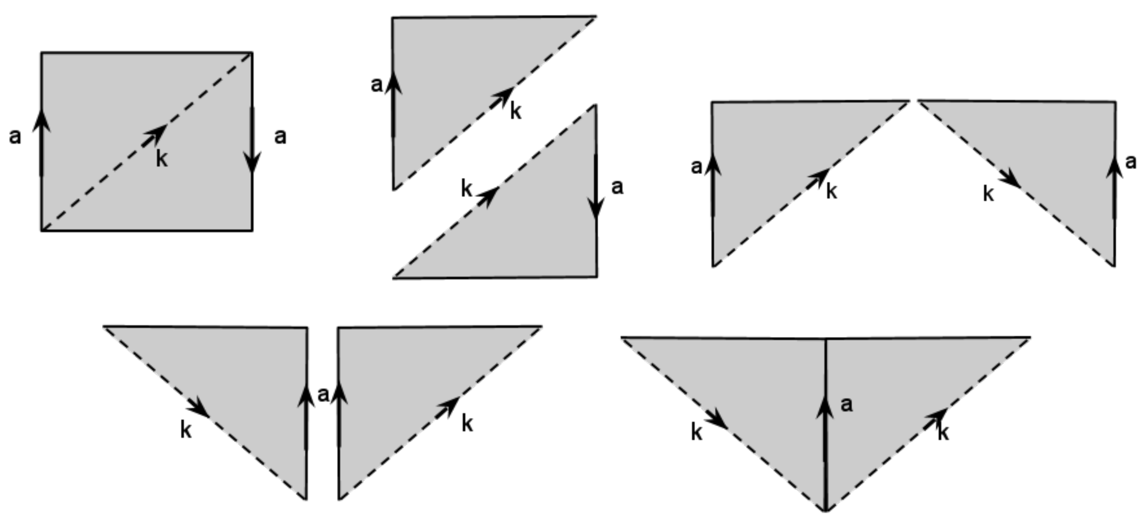


There might be many different diagrams that represent the same surface. For example, we could draw the diagram for the torus in the following ways (and this isn't even remotely all of them!). The important thing for a square to represent the torus is that opposite edges are identified without twists.

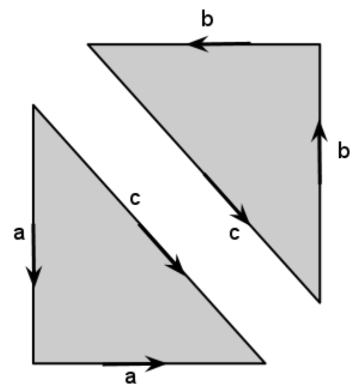


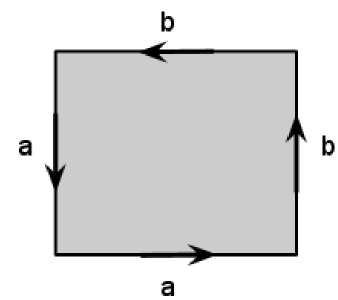
One technique for showing that two gluing diagrams represent the same surface is to take one of the diagrams, cut it, and reglue it (possibly repeatedly) until it looks like the other.

Example 7 (A Klein bottle is made from two Möbius strips)**.** In this example, we'll show that gluing two Möbius strips together along their boundary circles results in the Klein bottle. This explains the limerick at the beginning of these notes! First, we'll cut and rearrange the gluing diagram for the Möbius strip so that the boundary circle is displayed in one continuous piece.



Now we can see that the top edge of the triangle is the boundary of the Möbius strip, so this makes it easier to take two copies of the Möbius strip (in its new gluing diagram) and glue them together along their boundary circles (the boundary circles are labeled c in the diagram below on the left).



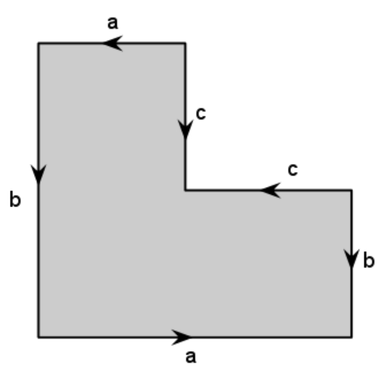


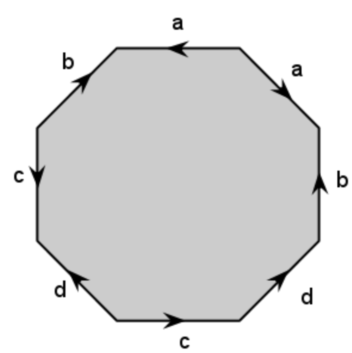
Hrmm.. This doesn't quite look like our standard diagram for the Klein bottle! Your job in the next problem is to figure out how to cut it and rearrange the pieces so that it looks like the standard diagram.

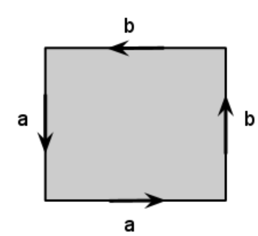
Problem 1. Use cutting and regluing techniques to show that the gluing square above right represents the Klein bottle. *Hint:* Cut along a diagonal.

Problem 2. What surface results from gluing a disk to the boundary circle of a Möbius band?

Problem 3. Which of the following diagrams represent equivalent surfaces? (Note that each diagram represents its own surface. It is not intended that you glue all the a's together, etc, but only the ones on that specific diagram.)

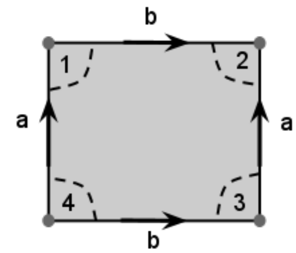






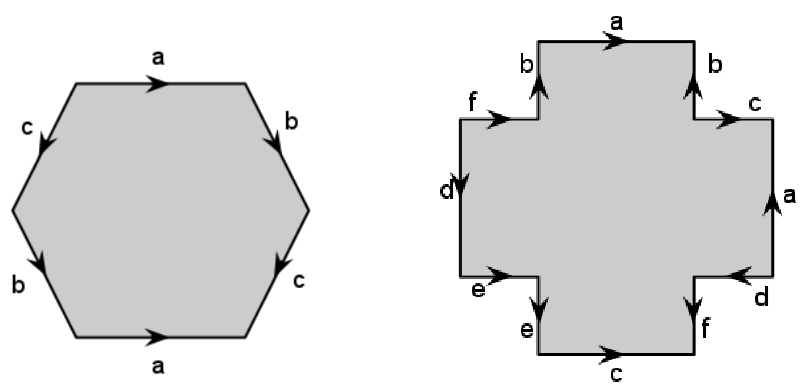
In a gluing diagram, we identify the edges of a polygon. This means that sometimes, the corners of our polygon are not distinct points on the surface it represents.

Problem 4. In the standard gluing diagram for the torus, all four corners represent the same point in on the surface of the torus. Cutting out a disk around this point is the same as cutting out the corners in the gluing diagram. Paste together the corners $1, 2, 3$, and 4 so they form a disk. Do the same for a Klein bottle. What happens for $\mathbb{R}P^2$?



Exercise 6. Which corners in the standard square diagram for the Klein bottle represent distinct points in the surface? What about in the standard square for $\mathbb{R}P^2$?

Exercise 7. In each of the following diagrams, identify which corners represent the same point and which are distinct.



Problem 5. Since we are topologists, we don't care so much whether lines are straight or curved. We could also think about gluing diagrams that result from dividing a circle into subsegments (edges) and assigning letters and arrows to these edges. Our example of the circular world map is a gluing diagram for the sphere S^2 as a circle divided into two edges. Find a similar diagram for $\mathbb{R}P^2$.

One way to record the gluing is by writing down a word that describes what letters we see when we walk around the edges of the gluing diagram. Begin at one corner of the diagram and walk around the perimeter of the diagram. When we walk along an edge labeled with a letter, say a, in the same direction as its assigned arrow, we write that letter. If we walk along an edge labeled with a letter, say a, but in the opposite direction of its assigned arrow, we write down a'. The string of letters contains the same information as the gluing diagram, so long as we remember the code that translates between the words and the gluing diagram.

Exercise 8. Draw the gluing diagrams associated with the following words: $abab$, $abca'b'c'$, $aba'b$, $ba'ba'$, $ab'ab$, $bacc'b'a$.

Problem 6. Do any of the words in the previous exercise represent the same surface?

Problem 7. Consider gluing diagrams for a square that glue together pairs of edges. Let's use the letters a and b to denote the pairs of edges. How many are there? *Hint:* To count them, you need to keep track of the letter of each edge and also its direction. Use the idea above of walking around the edge and recording the word you walk along. So, this is really a question that asks: how many four letter words are that use the letters a, a', b, b' such that both a and b appear exactly twice (where twice means with or without the decoration $'$, e.g. you could have a and a, or a and a', or a' and a' in your list, but you cannot have a appearing only once or three times).

Now back to our initial question of trying to list all surfaces that there are. We might start by trying to count the different surfaces represented by these gluing diagrams. The number we just arrived at is certainly too large. For example, if one diagram can be obtained from another by rotating it a quarter turn to the right, then these must represent the same surface. Similarly, if one diagram can be obtained from another by flipping the square over, they also must represent the same surface. By rotating and flipping our diagrams, we can reduce to the case where the left edge of the square is labelled with a and the arrow points up.

Problem 8. Now that we've determined that we can reduce to the case where the left edge of the square is labelled with a and with an upward pointing arrow, try to make a complete list of gluing diagrams that doesn't have any "obvious" repeats. By "obvious," I mean there isn't a sequence of rotations and a flip that will take one diagram on your list to another. Can you identify any of the diagrams as surfaces that we know?

Problem 9. Two of our diagrams turn out to represent the Klein bottle and two represent the projective plane $\mathbb{R}P^2$. Find a way to cut and paste the non-standard diagrams of the Klein bottle and $\mathbb{R}P^2$ so that they look like the standard ones.

You've just made a list of all the surfaces one can represent using a square, a lovely accomplishment! I hope you have enjoyed this brief journey into the twisted world of the topology of surfaces. You're already quite close to having all the tools necessary to make a list of *all* possible surfaces. If you'd like to do so, a good jumping off point from here is to look up "Classification of Surfaces" online or in one of the texts referenced below.