Ordinal arithmetic

Berkeley Math Circle

2018-3-5

Problem: How can we extend arithmetic to infinite numbers in an interesting way? The answer was found by Cantor: use ordinal arithmetic.

1 Definition of ordinals

Countable ordinals can be represented as subsets of the real line so that one can only make a finite number of leftward jumps in the subset. We only care about the order of points in the ordinal, so two subsets ordered in the "same way" count as the same ordinal. For example, $\{0, 1, 2\}$ and $\{-37, 3, \pi\}$ count as the same ordinal. Equivalently any nonempty subset of the ordinal has a smallest element.

Examples: 0, 1, 2, ... are subsets with 0, 1, 2, ... points. ω is the smallest infinite ordinal and can be represented as a subset 0,1/2, 2/3, 3/4,... with an infinite number of points.

2 Addition of ordinals

$\alpha + \beta$ is given by putting β to the right of α. So $\omega + 1$ might be represented by $0, 1/2, 2/3, 3/4, ..., 1$. Note that $1 + \omega = \omega \neq \omega + 1$ (Hilbert's hotel). In particular addition is not commutative, but is associative.

In general about half of the usual rules of arithmetic hold for ordinals.

3 Ordering of ordinals

We can compare α with β by matching their first elements, then their second elements, and so on. We find that either $\alpha < \beta$ (if α runs out first) or $\alpha = \beta$ or $\alpha > \beta$ (if β runs out first). Most of the usual rules hold for $<$ (but note that $1 + \omega = \omega$).

Limits. An increasing sequence of ordinals may get "closer and closer" to α: we say the sequence has limit α. For example $0, 1, 2, ...$ has limit ω. Continuity of functions can be defined as for real functions. Note that $\alpha + \beta$ is continuous in β but not in α: look at $n + 1$.

Subtraction. If $\alpha \geq \beta$ we can define $\alpha - \beta$. so that $\beta + (\alpha - \beta) = \alpha$.

4 Multiplication of ordinals

The product $\alpha\beta$ means "take β copies of α". It can also be defined by $\alpha.0 = 0$, $\alpha.(\beta+1) = \alpha, \beta+\alpha$, $\alpha\beta$ is continuous in β. (But $\alpha\beta$ is not continuous in α: look at $(\lim_{n\to\omega} n)\omega = \omega\omega \neq \omega = \lim_{n\to\omega}(n\omega)$.) Multiplication is associative but not commutative ($2\omega \neq \omega.2$). Example: $\omega\omega$ is the "lexicographic" order of points in a quadrant. Multiplication is sort of half distributive: $\alpha(\beta+\gamma) = \alpha\beta + \alpha\gamma$, but $(\omega+1)2 \neq \omega 2 + 1.2$.

Division-with-remainder: if $\beta > 0$ and α is an ordinal then $\alpha = \beta q + r$ for unique q, r with $r < b$.

Ordinals up to ω^ω are polynomials in ω with coefficients $0, 1, 2, ...$

5 Exponentiation

Exponentiation can be defined by: α^β continuous in β, $\alpha^{\beta+1} = \alpha^\beta.\alpha$, $\alpha^0 = 1$. α^β can be represented by functions from a finite subset of β to α, which can be thought of as words of length β to an alphabet α with most letters of the words missing 0, or as polynomials in α with coefficients $< \alpha$ and exponents in β. It is continuous in β but not continuous in α: $n^\omega = \omega$. $\alpha^{\beta+\gamma} = \alpha^\beta\alpha^\gamma$.

Ordinals have a base α expansion (if $\alpha > 1$). Cantor normal form is base ω expansion. Example: $(\omega^3.2+2)(\omega+5) = \omega^4 + \omega^3.15 + \omega + 10$.Some e

6 Prime ordinals

The prime ordinals are $2, 3, 5, ..., \omega, \omega+1$. $\omega+2 = (\omega+1)2$. ω^2+1, ω^3+1,...ω^ω, $\omega^\omega+1$, ...In general the primes are finite primes, ω^{ω^α} and $\omega^\alpha+1$.

Factorization in to primes need not be unique. $2\times 3 = 3\times 2$, $2\times\omega = \omega$, $\omega\times\omega^\omega = \omega^\omega$. Unique if (1) limits come first (2) limits in decreasing order (3) successors in decreasing order.

7 The ordinal ϵ_0

Problem with cantor normal form: ω^α is usually far bigger than α, but some ordinals may satisfy $\alpha = \omega^\alpha$ ϵ_0 is the limit of ω, ω^ω, ω^{ω^ω},.... and is the smallest ordinal that cannot be written using ω and 0 and addition and multiplication and exponentiation. Ordinals up to ϵ_0 can be represented as rooted trees.

ϵ^α is the α'th ordinal with $\alpha = \omega^\alpha$. More generally we get the Veblen hierarchy: $\phi_0(\alpha) = \omega^\alpha$. $\psi_\beta(\alpha)$ enumerates fixed points of ϕ_γ for γ_β. So $\epsilon_\alpha = \phi_1(\alpha$. Γ_0 is the smallest ordinal that cannot be written even using ϕ. There are also even bigger ordinals Γ_α.

Some even bigger ordinals: the Church-Kleene ordinal is the smallest that cannot be described in a computable (recursive) way. Far beyond this is the first uncountable ordinal $\aleph_1$.