Constructing Musical Scales

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Intervals

Definition 1. A *pure wave* is a sound wave of the form $y = A \sin(f \cdot (t + c))$, where A is the amplitude and f is the pitch of the wave.

Definition 2. An *interval* is a positive ratio r between two *pitches*.

For example, the interval from A_3 ($=$ 220 Hz) to E_4 ($\approx$ 330 Hz) is approximately $\frac{3}{2}$.

Definition 3. An *octave* is the interval 2.

Definition 4. A *perfect fifth* is the interval $\frac{3}{2}$.

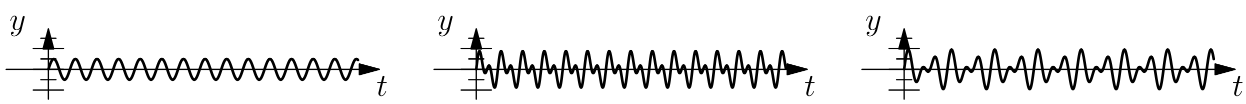

Figure 1: A pure sine wave, a pure sine wave played with the octave above it, and a pure sine wave played with the perfect fifth above it.

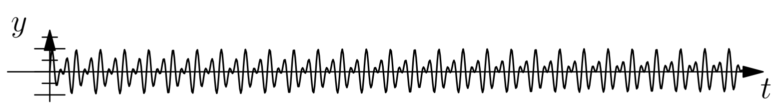

Figure 2: A perfect fifth in the 12-note scale. Although the waveform changes subtly over time, the change is too subtle for most of us to notice.

Exercise 1. Intuitively, when does an interval between two pitches sound "good"?

Beats

Consider the wave $\sin(f \cdot x) + \sin((f + \epsilon) \cdot x)$ with frequencies $\epsilon \ll f$. By the sum-to-product identities this becomes $\sin\left(\left(f + \frac{\epsilon}{2}\right) \cdot x\right) \cos\left(\frac{\epsilon}{2} x\right)$. Since ϵ is small, the period of $\cos\left(\frac{\epsilon}{2}\right)$ is much larger than the period of $\sin\left(\left(f + \frac{\epsilon}{2}\right) \cdot x\right)$; thus your ears hear a note with frequency $f + \frac{\epsilon}{2}$ fading in and out every time $\cos\left(\frac{\epsilon}{2} x\right) = 0$, which occurs every $\frac{1}{\epsilon}$ seconds.

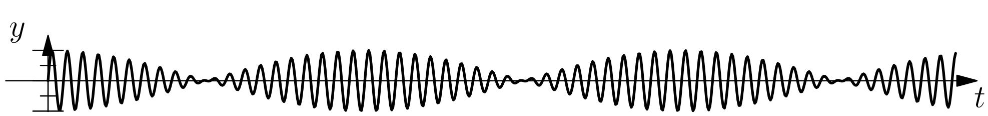

Figure 3: Two close pure tones played together.

Overtones

Definition 5. The n^{th} *harmonic* and the $(n-1)^{th}$ *overtone* is the interval n.

When you hear a note from a musical instrument, you do not hear a pure tone. You hear a series of overtones. Figure 4 shows how much of each overtone is produced in the *attack* (beginning) of the A_3 220 Hz note, played on various instruments.

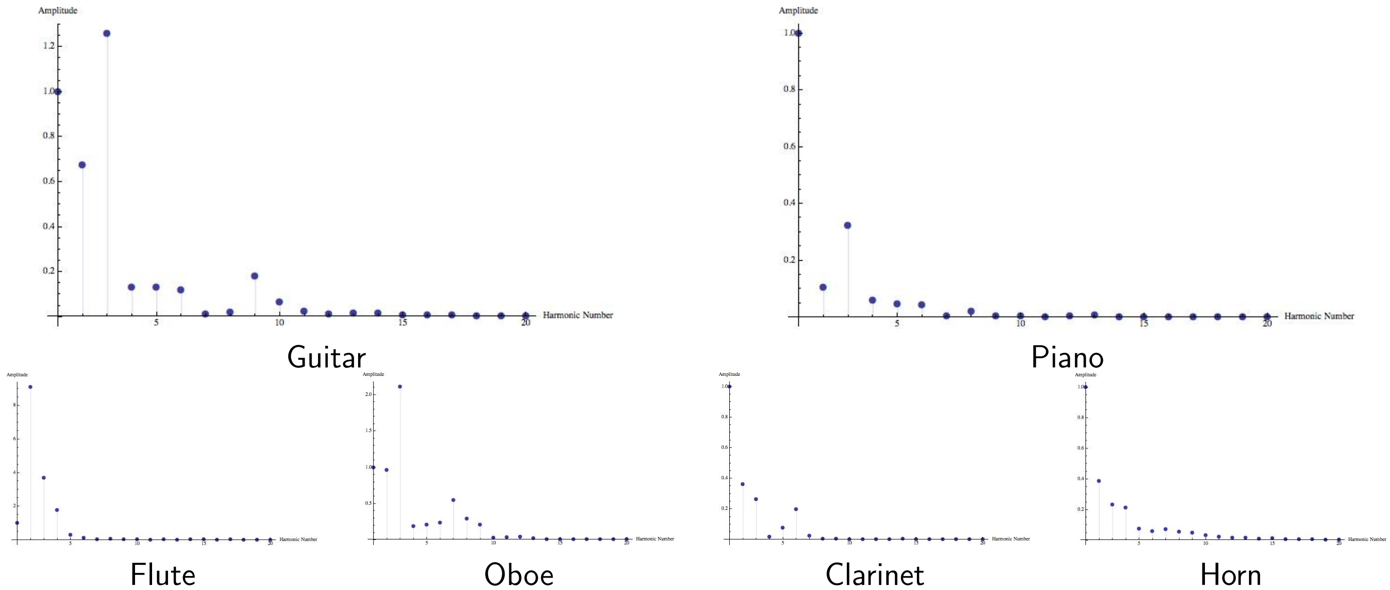

Figure 4: Overtone series of various instruments [1]

When the interval $\frac{m}{n}$ with positive integers m, n small is played on an instrument, not only is the period of the resulting waveform small, but some of the harmonics line up. In addition, when playing an interval approximating but not exactly equal to $\frac{m}{n}$, some harmonics will interfere with each other.

Exercise 2. The tones D_4 and A_4 oscillate at approximately $293.66\,\mathrm{Hz}$ and $440\,\mathrm{Hz}$, respectively. What frequency of beats produced by the 3rd harmonic of D_4 and the second harmonic of A_4?

Scales

An initial approach

Definition 6. A set $\mathcal{S}$ is a $\mathcal{F}$*-scale*, where $\mathcal{S}, \mathcal{F}$ are sets of positive real frequencies that satisfies the following properties:

1. $\mathcal{F} \subset \mathcal{S}$,
2. $\forall s \in \mathcal{S},\ 2s, \dfrac{s}{2} \in \mathcal{S}$
3. $\forall f_1, f_2, f_3 \in \mathcal{S},\ \dfrac{f_1 f_2}{f_3} \in \mathcal{S}$,
4. $\mathcal{S}$ is not dense at any positive real number.

Exercise 3. Find the smallest $\{220\,\mathrm{Hz}\}$-scale.

Exercise 4. Find the smallest $\{220\,\mathrm{Hz}, 330\,\mathrm{Hz}\}$-scale.

Take 2

Definition 7. A set $\mathcal{S}$ is a *$\mathcal{F}$-scale* with error $\delta > 0$, where $\mathcal{S}, \mathcal{F}$ are sets of positive real frequencies that satisfies the following properties:

1. $\mathcal{F} \subset \mathcal{S}$,
2. $\forall s \in \mathcal{S},\ 2s, \dfrac{s}{2} \in \mathcal{S}$
3. $\forall f_1, f_2, f_3 \in \mathcal{S},\ \left(\dfrac{1}{1+\delta} \cdot \dfrac{f_1 f_2}{f_3}, (1+\delta) \cdot \dfrac{f_1 f_2}{f_3}\right) \cap \mathcal{S} \neq \emptyset$
4. $\mathcal{S}$ is not dense at any positive real number.

Exercise 5. Find a $\{220\,\mathrm{Hz}, 330\,\mathrm{Hz}\}$-scale with error 10% by finding a power of $\frac{3}{2}$ within a factor of 1.1 from a power of 2.

Exercise 6. Repeat exercise 5 with error 2%.

Definition 8. A *perfect major third* is the interval $\frac{5}{4}$.

Exercise 7. Complete one octave of intervals in the incomplete $\left\{1, \frac{3}{2}, \frac{5}{4}\right\}$-scale shown in figure 5, using denominators ≤ 10 for all but two notes.

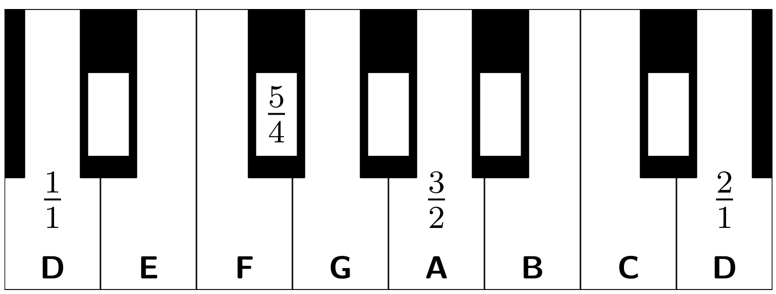

Figure 5: One octave starting from D, with some intervals noted

Modern Approach: Equal Temperament

Definition 9. A set $\mathcal{S}$ is an *equal tempered $\mathcal{F}$-scale* with error $\delta > 0$ if

1. $\mathcal{S}$ is of the form $k \cdot 2^{\mathbb{Z}/n}$, where $n \in \mathbb{N}$ and $2^{\mathbb{Z}/n} = \left\{2^{m/n} \;\middle|\; m \in \mathbb{Z}\right\}$ ($\mathcal{S}$ is said to have n notes per octave and base frequency k),
2. $\forall f \in \mathcal{F},\ \left(\dfrac{1}{1+\delta} f, (1+\delta) f\right) \cap \mathcal{S} \neq \emptyset.$

Exercise 8. Find the infinum (roughly speaking, *minimum*) error of the equal-tempered $\left\{1, \frac{3}{2}\right\}$-scale with base frequency 1 and 12 notes per octave.

Exercise 9. Find the infinum error of the equal-tempered $\left\{1, \frac{3}{2}, \frac{5}{4}\right\}$-scale with base frequency 1 and 12 notes per octave.

Exercise 10. Find the infinum error of the equal-tempered $\mathcal{F}$-scale, where $\mathcal{F}$ is the set of intervals you found in Excercise 7 / Figure 5.

Finding good equal-tempered microtonal scales for the perfect fifth

Definition 10. *Rational approximation* is the process of finding rational numbers that deviate from an irrational number by a small distance for some metric.

Exercise 11. Find an irrational number such that its rational approximations will yield good equal-tempered $\left\{1, \frac{3}{2}\right\}$-scales with base frequency 1. (1)

Definition 11. The *continued fraction* for an irrational number a is the unique fraction of the form

$$a = a_0 + \cfrac{1}{a_1 + \cfrac{1}{a_2 + \cfrac{1}{a_3 + \cfrac{1}{\ddots}}}}$$

for integer a_0 and positive integers $a_1, a_2, a_3, \ldots$.

Exercise 12. Work out the first few terms of the continued fraction for your answer to exercise 11.

The continued fraction is often used for rational approximation by stopping the fraction after a couple of layers.

Exercise 13. Use the continued fraction you found in exercise 12 to find at least two equal-tempered scales with more than 12 notes.

Finding good equal-tempered microtonal scales for more intervals [2]

Exercise 14. Think about why approximating the first few terms of $\mathbb{N}$ in an equal-tempered scale will give "good"-sounding intervals (see exercise 1).

Exercise 15. Find a trigonometric function in terms of n and N that peaks when N is well-approximated by a note in the equal-tempered n-note scale with base frequency 1. (2)

Exercise 16. Write out the first few terms of

$$\sum_{N=1}^{\infty} \frac{1}{N^a} f_N(n),$$

where $a > 1$ and $f_N(n)$ is the function you found in exercise 14.

Exercise 17. Interpret what it means musically for the sum in exercise 16 to be high.

Exercise 18. What is the role of the parameter a in exercise 16? What do high and low a mean?

Exercise 19. Simplify the expression in exercise 16 using the substitution $\cos(x) = \Re\left(e^{ix}\right)$.

Definition 12. For $\Re(s) > 1$, the *Riemann zeta function* is $\zeta(s) = \sum_{N=1}^{\infty} \frac{1}{N^s}$.

Exercise 20. Write the expression from exercise 19 in the form $\Re\left(\zeta(s)\right)$ for some s in terms of n and a.

Exercise 21. Find some scales which our approach with continued fractions in exercise 13 failed to discover.

Hints:

(1) $2^{m/n} \approx \frac{3}{2}$
(2) For what values of x does $\cos(2\pi x)$ peak?

References

[1] Maria Bell. *Fourier Analysis in Music*. URL: https://www.projectrhea.org/rhea/index.php/Fourier_analysis_in_Music.

[2] genewardsmith. *The Riemann Zeta Function and Tuning*. URL: https://xenharmonic.wikispaces.com/The+Riemann+Zeta+Function+and+Tuning.