Berkeley Math Circle: A Taste of Asian Math Olympiads

Intermediate I - In Number Theory
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I. Useful Knowledge

- *Viejta's Formula*
  If a quadratic equation $ax^2 + bx + c = 0$ has two solutions x_1 and x_2 (not necessarily distinct), then
  $$x_1 + x_2 = \frac{-b}{a}$$
  and
  $$x_1 x_2 = \frac{c}{a}$$
- *Rational Root Theorem*
  If a polynomial $\sum_{i=0}^{n} a_i x^i$ $(a_i \in \mathbb{Z}\ \forall i)$ has a rational root, then that root is of the form $\frac{c}{d}$ with $c|a_0$ and $d|a_n$.
- *Introductory number theory and Diophantine equations.* Brief understanding of modular arithmetics (congruences *modulo* n) is helpful, but strong familiarity with divisbility rules and arguments using remainders (when dividing by n) is sufficient.
- Strong background in algebraic manipulations.

II. Problems

The following problems are taken from various high school entrance preparation exams to the *Specialty High School of National University of Vietnam, HCMC* along with other sources.

1. **Warm-Ups** When is the sum of 2 numbers odd? When is it even? What about the sum of n numbers?
   When is $1 + 2 + ... + n$ an odd number? When is it even?

   Try to find a pattern. Can you prove it?

What about the sum of any n consecutive numbers?

2. If a and b are not divisble by 3, when is $a+b$ divisible by 3?

   If we have 10 numbers, each yields remainder 1 when dividing by 3, then what is the remainder of their sum? what about their product?

   When will the sum of n numbers, each with remainder 1 when dividing by 3, be divisible by 3?

3. *** Let $p_1 < p_2 < ... < p_{17}$ be primes such that $\sum_{i=1}^{17} p_i^2$ is a perfect square. Show that $(p_{17}^2 - p_{16}^2)$ is divisible by p_1.

4. **Split $1, 2, 3, ..., 2n$ into two groups: $a_1 < a_2 < a_3 < ... < a_n$ and $b_n < b_{n-1} < ... < b_1$. Show that

$$\sum_{i=1}^{n} |a_i - b_i| = n^2$$

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5. Equation $x^2+ax+b=0$ has two integer solutions. Know $3a+b=8$. Find the two integer solutions.

6. Given $m, n \in \mathbb{N}$. If $(5n+m)|(5m+n)$, deduce that $n|m$.

7. Let S be the set of all ordered triples (p, q, r) of prime numbers for which the equation $px^2+qx^2+r=0$ has at least one rational solution. How many primes appear in S at least seven times?