CONVEXITY AND ITS APPLICATIONS TO INEQUALITIES

1. Basics

Definition: A function $f(x)$ is called *convex* on an interval (a, b) if

$$\frac{f(x)+f(y)}{2} \geq f\left(\frac{x+y}{2}\right) \tag{1.1}$$

holds for all numbers x, y from (a, b). If the opposite inequality holds in (1.1) then the function f is called *concave*.

Fact 1. The function[1] $f(x)$ is convex on (a, b) if and only if $f''(x) \geq 0$ for all x from (a, b) (if the opposite inequality holds, i.e., $f''(x) \leq 0$ then f is concave).

Fact 2. $f(x)$ is *increasing* on (a, b), i.e., $f(x_2) > f(x_1)$ for all $x_2 > x_1$ from (a, b), if and only if $f'(x) > 0$ for all x in (a, b). $f(x)$ is *decreasing* on (a, b), i.e., $f(x_2) < f(x_1)$ for all $x_2 > x_1$ from (a, b), if and only if $f'(x) < 0$ for all x in (a, b).

Exercise 1. Show that if $p > 0$ then $f(x) = x^p$ is increasing for $x \geq 0$.

Exercise 2. Show that $f(x) = x^p$ is convex for $x > 0$ if $p \geq 1$, and it is concave for $x > 0$ if $0 < p \leq 1$. What happens when $p < 0$?

Problem 1. Assume $f(x)$ is convex on (a, b). Then show that for any real number p in (a, b) we have $f(x) \geq f(p) + f'(p)(x - p)$ for all x (this problem is known as *the convex function lies above its tangent line*).

Problem 2. If $f(x)$ is convex on (a, b) show that for any real number p from (a, b) the function

$$\varphi(t) = f(p+t) + f(p-t)$$

is increasing for $t \geq 0$ while both points $p + t$ and $p - t$ remain in the interval (a, b) (this is the simplest version of *Karamata's inequality*). What happens when f is concave?

Problem 3. If $f(x)$ is convex on (a, b) then show that for any numbers $x_1, \ldots, x_n$ from (a, b), and any nonnegative numbers $\alpha_1, \ldots, \alpha_n \geq 0$ with $\alpha_1 + \ldots + \alpha_n = 1$ we have

$$\alpha_1 f(x_1) + \ldots + \alpha_n f(x_n) \geq f(\alpha_1 x_1 + \ldots + \alpha_n x_n)$$

(*Jensen's inequality*). What happens if f is concave?

Problem 4*. If $f(x)$ is convex for $x > 0$ then show that for any positive numbers $x_1, \ldots, x_n, y_1, \ldots, y_n > 0$ we have

$$B(x_1, y_1) + \ldots + B(x_n, y_n) \geq B(x_1 + \ldots + x_n, y_1 + \ldots + y_n).$$

where $B(x, y) = y f(x/y)$. What happens when f is concave? (Hint: first show the inequality when $n = 2$)

[1] In what follows we are assuming that all our functions $f(x)$ are twice continuously differentiable so that we can write the derivatives of f without thinking if such exist

2. Some olympiad problems

Problem 1. Let $a, b, c > 0$ be such that $a^2 + b^2 + c^2 = 3$. Show that

$$\frac{1}{a^3+2} + \frac{1}{b^3+2} + \frac{1}{c^3+2} \geq 1.$$

Problem 2. Let $a, b, c, d, e \geq 0$ be such that

$$\frac{1}{4+a} + \frac{1}{4+b} + \frac{1}{4+c} + \frac{1}{4+d} = 1.$$

Show that

$$\frac{a}{4+a^2} + \frac{b}{4+b^2} + \frac{c}{4+c^2} + \frac{d}{4+d^2} + \frac{e}{4+e^2} \leq 1.$$

Problem 3. Let a, b, c be positive numbers so that $a + b + c = 1$. Prove

$$10(a^3 + b^3 + c^3) - 9(a^5 + b^5 + c^5) \geq 1.$$

Problem 4. Let a, b, c be positive numbers such that $a^2 + b^2 + c^2 = 12$. Find the maximal possible value of

$$a(b^2+c^2)^{1/3} + b(c^2+a^2)^{1/3} + c(a^2+b^2)^{1/3}.$$

Problem 5. For any nonnegative $x_1, \dots, x_n \geq 0$ with $\sum_{k=1}^n x_k = 1$ show that

$$\sum_{k=1}^{n} x_k(1-x_k)^2 \leq \left(1 - \frac{1}{n}\right)^2.$$

Problem 6. Let a, b, c be positive real numbers. Prove that

$$\frac{a}{\sqrt{a^2+8bc}} + \frac{b}{b^2+8ca} + \frac{c}{c^2+8ab} \geq 1.$$

Problem 7. Let a, b, c be positive numbers. Show that

$$\frac{(b+c-a)^2}{a+(b+c)^2} + \frac{(a+b-c)^2}{c+(a+b)^2} + \frac{(a+c-b)^2}{b+(a+c)^2} \geq \frac{3}{5}.$$

Problem 8 Let a, b, c be positive real numbers such that $abc = 1$. Prove that

$$\frac{1}{a^3(b+c)} + \frac{1}{b^3(c+a)} + \frac{1}{c^3(a+b)} \geq \frac{3}{2}.$$

Problem 9. For any $a, b, c > 0$ prove

$$\frac{a}{b+c} + \frac{b}{c+a} + \frac{c}{a+b} \geq \frac{3}{2}.$$

Problem 10. Prove for all reals $a, b, c \geq 0$:

$$\frac{(a+b+c)^2}{3} \geq a\sqrt{bc} + b\sqrt{ac} + c\sqrt{ab}.$$

Problem 11. Prove for all positive real numbers a, b, c:

$$\frac{9}{a+b+c} \leq 2\left(\frac{1}{a+b} + \frac{1}{b+c} + \frac{1}{c+a}\right).$$

3. Some classical inequalities

Problem 1. Show that for any nonnegative numbers $x_1, \dots, x_n$ we have

$$\frac{x_1 + \ldots + x_n}{n} \geq (x_1 \cdots x_n)^{1/n} \qquad (\textit{AM-GM: arithmetic-geometric mean inequality}).$$

Problem 2. Show that for any $x, y \geq 0$ and any $\alpha, \beta \geq 0$ with $\alpha + \beta = 1$ we have

$$\alpha x + \beta y \geq x^{\alpha} y^{\beta} \qquad (\textit{Young's inequality}).$$

Problem 3. Show that if $\alpha \geq 1$ and $x \geq -1$ then

$$(1+x)^{\alpha} \geq 1 + \alpha x \qquad (\textit{Bernoulli's inequality}),$$

while for $0 \leq \alpha \leq 1$ the opposite inequality holds.

Problem 4. Show that for any nonnegative numbers $x_1, \dots, x_n, y_1, \dots, y_n \geq 0$ we have

$$\left(\sum_{j=1}^{n} x_j y_j\right)^2 \leq \left(\sum_{j=1}^{n} x_j^2\right)\left(\sum_{j=1}^{n} y_j^2\right) \qquad (\textit{Cauchy–Schwarz inequality}).$$

Problem 5. Show that for any positive numbers $p, q > 0$ with $\frac{1}{p} + \frac{1}{q} = 1$, and any nonnegative numbers $x_1, \dots, x_n, y_1, \dots, y_n \geq 0$ we have

$$\sum_{j=1}^{n} x_j y_j \leq \left(\sum_{j=1}^{n} x_j^p\right)^{1/p} \left(\sum_{j=1}^{n} y_j^q\right)^{1/q} \qquad (\textit{Hölder's inequality}).$$

Problem 6. Show that for any $1 \leq p < \infty$, and any positive numbers $x_1, \dots, x_n, y_1, \dots, y_n \geq 0$ we have

$$\left(\sum_{j=1}^{n} (x_j + y_j)^p\right)^{1/p} \leq \left(\sum_{j=1}^{n} x_j^p\right)^{1/p} + \left(\sum_{j=1}^{n} y_j^p\right)^{1/p} \qquad (\textit{Minkowski inequality}).$$

Problem 7. Let $x_1, \dots, x_n$ be positive numbers. Show that the following function

$$f(p) = \left(\sum_{j=1}^{n} x_j^p\right)^{1/p}$$

is decreasing for $p \geq 1$, and increasing for $0 < p \leq 1$.

Problem 8* Let $x_1, \dots, x_n$ be positive numbers. Show that the following function

$$f(p) = \left(\frac{\sum_{j=1}^{n} x_j^p}{n}\right)^{1/p} \qquad \text{is nondecreasing.}$$

Problem 9*. Show that for any $2 \leq p < \infty$, and any positive numbers $x_1, \dots, x_n, y_1, \dots, y_n \geq 0$ we have

$$\sum_{j=1}^{n} (x_j + y_j)^p + \sum_{j=1}^{n} |x_j - y_j|^p \leq$$

$$\left(\left(\sum_{j=1}^{n} x_j^p\right)^{1/p} + \left(\sum_{j=1}^{n} y_j^p\right)^{1/p}\right)^p + \left|\left(\sum_{j=1}^{n} x_j^p\right)^{1/p} - \left(\sum_{j=1}^{n} y_j^p\right)^{1/p}\right|^p \qquad (\textit{Hanner's inequality}).$$

4. Problems that originate from a research

I am not assuming that one should solve all these problems, however, one can try to solve some particular cases.

Problem 0. Show that

$$x^{3/2} - \frac{1}{\sqrt{2}}(2x - \sqrt{x^2+y^2})\sqrt{x+\sqrt{x^2+y^2}} \leq \frac{3}{8}\frac{y^2}{\sqrt{x}} \quad \text{for all} \quad x, y \geq 0.$$

(*Improving Beckner's bound*).

Problem 1. For any positive numbers a, b show that

$$\frac{a\ln a + b\ln b}{2} - \left(\frac{a+b}{2}\right)\ln\left(\frac{a+b}{2}\right) \leq \frac{(a-b)^2}{16}\left(\frac{1}{a}+\frac{1}{b}\right)$$

(*Log-Sobolev inequality*).

Problem 2*. Let $1 < p \leq q < \infty$. Show that

$$\left(\frac{\left|a+\sqrt{\frac{p-1}{q-1}}\right|^q + \left|a-\sqrt{\frac{p-1}{q-1}}\right|^q}{2}\right)^{1/q} \leq \left(\frac{|a+1|^p + |a-1|^p}{2}\right)^{1/p} \quad \text{holds for all real } a$$

(*Hypercontractivity*). Try $p = 2$ and $q = 4$.

Problem 3. Let $1 \leq p \leq 2$. Show that for all $0 \leq a \leq 1$ we have

$$a^2 + (p-1) \leq \left(\frac{(1+a)^p + (1-a)^p}{2}\right)^{2/p}$$

(*Hausdorff–Young inequality: a simplified version*).

Problem 4. Let real numbers $x_1, \ldots, x_n$ be such that

$$\frac{1}{n}\sum_{j=1}^{n} x_j = 1.$$

Show that

$$\frac{1}{n}\sum_{j=1}^{n} e^{-x_j^2/n} \leq e^{-1/n}$$

(*Chang–Wilson–Wolff's superexponential bound in arbitrary dimensions*).

Problem 5. For all x from $[0,1]$ and all integers $0 \leq k \leq n$ we have

$$(2 + 2x^k - 4x^n + 2x^{2n-k})^n \leq (2 - x^k)^{2n-k}$$

(*a mathoverflow question*).

Problem 6. For any numbers $1 \leq s \leq \lambda$, and any $p \geq 1$ show that

$$\frac{\lambda^p - 1}{\lambda^p - \lambda}(s^p - s) \leq s^p - 1.$$

(*Lower bounds for Hardy–Littlewood maximal functions*).

Problem 7*. Find the largest power $p > 0$ such that

$$(a+b+c)^p \leq (1+a^p)(1+b^p)(1+c^p)$$

holds for all nonnegative numbers $a, b, c \geq 0$. (*Kane–Tao: a problem about efficient clustering*).