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Polytopes and their graphs

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Definition. A set S in $\mathbb{R}^2$ (or more generally, $\mathbb{R}^d$) is called *convex* if the segment between any two points of S lies entirely in S. A *polytope* is the convex hull (=smallest convex set containing) of finitely many points. Its *graph* is its 1-dimensional structure, consisting of vertices (0-dimensional) and edges (1-dimensional). A graph is called *k-connected* if it has at least $k+1$ vertices, and between any two vertices there are (at least) k vertex-disjoint paths. The *distance* of two vertices a and b in a graph is the number of edges of some shortest path joining a and b. The *diameter* of a graph is the largest distance between two vertices in it.

Menger's theorem Let G be a graph with at least $k+1$ vertices. Then G is k-connected if and only if deleting less than k points from it, however chosen, we always get a connected graph.

Balinski's theorem The graph of every d-polytope is d-connected.

Steinitz's theorem A graph is 3-connected and planar $\iff$ it is the graph of some 3-polytope.

Examples.

1. Is the intersection of any two convex sets convex?
2. Given some subset A of $\mathbb{R}^2$ (possibly infinite), is there always a convex set containing A? Is there always a smallest one among all convex sets containing A?
3. Is the union of two convex subsets convex?
4. Given two vertices a and b of the cube, how many vertex-disjoint paths are there from a to b? What's the diameter of the graph of the cube?
5. For any positive integer d, construct a graph that is d-connected but not $(d+1)$-connected.
6. The converse of Balinski's theorem is false: Construct a graph that is 2-connected, not 3-connected, and not the graph of any 2-polytope.
7. Can you color the vertices of any 3-polytope using only 3 colors, so that no edge joins two vertices of the same color?
8. Can you draw the graph of any 3-polytope with straight edges in a polygonal portion of the plane, so that no two edges intersect in the middle?
9. (longer) Prove Menger's theorem.
10. Example of an open problem: We don't know of a theorem that ends with "... if and only if G is the graph of a 4-polytope". We also don't know if the diameter of the graph of any 4-polytope with N facets is at most N.