Problems with the Pigeon-Hole Principle

The principle states that if $n+1$ objects are split into n categories then there should be a category that contains at least two objects.

Some warm up problems

A. Among any six integers there are two whose difference is divisible by five.

B. Let $1 \leq a_1 < a_2 < \ldots < a_{n+1} \leq 2n$ be $n+1$ distinct integers between 1 and $2n$. Then there are two among these (possible the same number) whose sum equals a third one (so $a_i + a_j = a_k$ with the possibility that maybe $a_i = a_j$)

C. The interior of an equilateral triangle with side length 2 contains five points. Show that among these there are always two whose distance is at most 1.

1. Application to Number Theory.
Prove that for each real number $\theta \in [0,1]$ and each $N \in \mathbb{N}$ there exist positive integers $0 \leq a \leq q \leq N$ so that

$$|\theta - \frac{a}{q}| \leq \frac{1}{Nq}.$$

Comment: This easily implies that the inequality

$$|\theta - \frac{a}{q}| \leq \frac{1}{q^2}$$

has infinitely many integral solutions (a, q).

2. Application to Geometry
Consider $2n+1$ real numbers in the interval $[1, 2^n)$. Show that you can always select three among them to construct a triangle whose side lengths equal these numbers.

3. A combinatorial Application
Find the largest possible size N of a collection of distinct positive (> 0) numbers having the property that among any seven of them one can always find two whose product is 1.

4. Another combinatorial Application
Every point in the plane is colored either red, green, or blue. Prove that there exists a rectangle in the plane such that all four of its vertices have the same color.

Elementary geometry

5. The angle $\angle BAC$ in the triangle ABC measures 120 degrees. Let AD, BE and CF be the three angle bisectors. Prove that $\angle FDE$ is a right angle.

Problems with the AM-GM inequality

For each positive real numbers $a_1, a_2, \ldots, a_n$ the following inequality holds true

$$\sqrt[n]{a_1 a_2 \ldots a_n} \leq \frac{a_1 + a_2 + \ldots + a_n}{n}$$

6. Find the smallest possible value of the expression

$$\frac{x}{y} + (\frac{y}{z})^{1/2} + (\frac{z}{x})^{1/3}$$

where x, y, z range through positive real numbers.

7. Find the floor function value (integer part) of the expression

$$\sqrt{2} + \sqrt[3]{\frac{3}{2}} + \sqrt[4]{\frac{4}{3}} + \ldots + \sqrt[n+1]{\frac{n+1}{n}}$$

Elementary complex analysis

8. Consider the polynomial

$$P(z) = z^n + a_{n-1}z^{n-1} + \ldots + a_1 z + a_0$$

with $a_i \in \mathbb{C}$. Prove that there exists $|z| = 1$ with $|f(z)| \geq 1$.