EQUILATERAL TRIANGLES PROBLEM SET

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1. Some other ways to solve 2017 AIME I #15

Problems[1] one and two both employ the following setup. Suppose given $\triangle ABC$, with $BC = 3$, $CA = 4$, and $AB = 5$. Suppose equilateral $\triangle XYZ$ has vertices on $\triangle ABC$, with X on CA, Z on AB, and Y on BC. Let $\theta = \angle CXY$ and let s be the length of a side of $\triangle XYZ$.

1. Use the Law of Sines to find AZ and BZ in terms of s and θ. Then

- Find s in terms of θ.
- Find the minimum possible area of $\triangle XYZ$.

2 (Jonathan Kane's solution)**.** Place the 3-4-5 triangle on a coordinate system, so that the coordinates of vertices are $C = (0, 0)$, $A = (4, 0)$, and $B = (0, 3)$. Find the coordinates of Z in terms of s and θ, and then proceed as above.

Given a triangle, an *inner* triangle is a triangle with vertices on the sides of the given triangle and an *outer* triangle is a triangle with sides on the vertices of the given triangle.

3. Given an equilateral $\triangle XYZ$ with sides of length 2, suppose $\triangle ABC$ is an outer triangle of $\triangle XYZ$ which is similar to a 3-4-5 triangle. Suppose angle C is right, and and that C and X lie on opposite sides of line YZ. Let $\theta = \angle CYZ$ and use the Law of Sines to find AX and BX in terms of θ. Then

- Find the perimeter of $\triangle ABC$ in terms of θ.
- Find the maximum possible area of $\triangle ABC$.
- Given a 3-4-5 triangle ABC and an equilateral triangle DEF, let Inner$\triangle$ be the smallest inner equilateral triangle of $\triangle ABC$ and let Outer$\triangle$ be the largest outer triangle of $\triangle DEF$ which is similar to a 3-4-5 triangle. Show that $\frac{[\text{Inner}\triangle]}{[ABC]} = \frac{[DEF]}{[\text{Outer}\triangle]}$.

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[1] All solutions on this page are for my $3, 4, 5$ version of 2017 AIME I #15.

2. Areas of Equilateral Triangles: More Problems

4. Given three coplanar parallel lines such that the middle line is 6 units from the lines on either side of it, find the areas of the largest and smallest triangles similar to a 3-4-5 triangle and such that one vertex of the triangle lies on each line.

5. Suppose given two equilateral triangles $\triangle ABC$ and $\triangle DEF$ and points P and Q in the interiors of $\triangle ABC$ and $\triangle DEF$, respectively, such that $PA = QD$, $PB = QE$, and $PC = QF$. Prove that the triangles are congruent. Now repeat this problem, with P and Q in the exteriors of $\triangle ABC$ and $\triangle DEF$, respectively.

6. Find the area of the smallest equilateral triangle with vertices on concentric circles of radii 3, 4, and 5.

7. Prove that the circumcircles of equilateral triangles built on the exterior sides of a given triangle are concurrent.

8. Given concentric circles of radii a, b, and c.

- Prove that there is an equilateral triangle with one vertex on each circle if and only if there is a triangle with sides of length a, b, and c. Furthermore, up to congruence there are exactly two such triangles.
- Prove that the area of the largest equilateral triangle with one vertex on each circle is
$$\frac{3}{2}[ABC] + \frac{\sqrt{3}}{8}(a^2 + b^2 + c^2),$$
where $\triangle ABC$ has sides of length a, b, and c.
- What is the area of the smallest equilateral triangle with one vertex on each circle?

9. Prove that the area of the largest outer equilateral triangle of a given $\triangle ABC$ is

$$2[ABC] + \frac{\sqrt{3}}{6}(a^2 + b^2 + c^2)$$

and that the area of the smallest inner equilateral triangle of $\triangle ABC$ is

$$\frac{[ABC]^2}{2[ABC] + \frac{\sqrt{3}}{6}(a^2 + b^2 + c^2)}$$

10. Given $\triangle ABC$ with sides of length a, b, and c, construct the following.

- The largest equilateral triangle with one vertex on each of three concentric circles of radii a, b, and c.
- The smallest equilateral triangle with one vertex on each of three concentric circles of radii a, b, and c.
- The largest outer equilateral triangle of $\triangle ABC$.
- The smallest inner equilateral triangle of $\triangle ABC$.

11. Given non-congruent $\triangle ABC$ and $\triangle DEF$ such that $AB \parallel DE$, $BC \parallel EF$, and $CA \parallel FD$, prove that $\overleftrightarrow{AD}$, $\overleftrightarrow{BE}$, and $\overleftrightarrow{CF}$ are concurrent.

12. Given $\triangle ABC$, show that any two outer equilateral triangles are related by a spiral similarity centered at the Fermat point (or first isogonal center) of the triangle.

13. An affine transformation $\mathbb{R}^2 \to \mathbb{R}^2$ has the form $t(\vec{x}) = A\vec{x} + \vec{b}$, with A a 2×2 invertible matrix, and $\vec{b} \in \mathbb{R}^2$. Prove the following.

- An affine transformation sends lines to lines.
- An affine transformation sends parallel lines to parallel lines.
- An affine transformation preserves ratios of areas.
- There is an affine transformation sending any one triangle to any other.

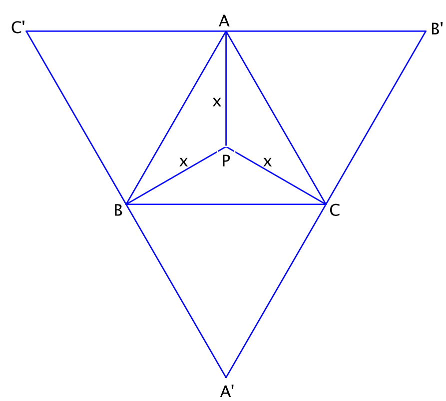
14. Given $\triangle ABC$, under what conditions are there two congruent inner equilateral triangles with pairwise parallel sides? Can you construct these triangles? Can you calculate the area of these triangles, if the sides of $\triangle ABC$ have lengths 3, 4, and 5?

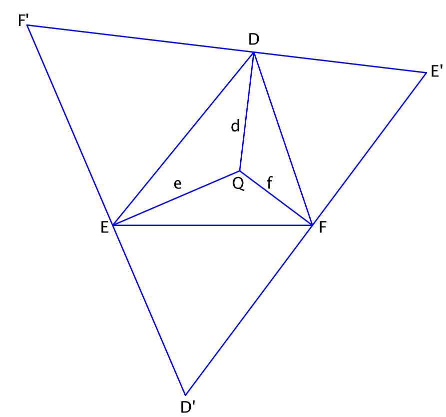
15. Given $\triangle ABC$ and $\triangle XYZ$ find formulas for the areas of the largest outer and smallest inner triangles of $\triangle ABC$ which are similar to $\triangle XYZ$. Can you construct these triangles?

3. An Inequality

16. The ratio of the area of a triangle to the area of its largest outer equilateral triangle is less than or equal to $\frac{1}{4}$, with equality only if the triangle is equilateral. Prove this as follows.

Suppose given $\triangle ABC$ and $\triangle DEF$, with $[ABC] = [DEF]$ and $\triangle ABC$ equilateral. Let P and Q be the Fermat points (or isogonic centers) of $\triangle ABC$ and $\triangle DEF$, respectively. Let $PA = PB = PC = x$ and let $d = QD$, $e = QE$, and $f = QF$. Finally, let $\triangle A'B'C'$ and $\triangle D'E'F'$ be the largest outer equilateral triangles of $\triangle ABC$ and $\triangle DEF$, respectively, and let H_1 and H_2 be the respective altitudes of $\triangle A'B'C'$ and $\triangle D'E'F'$. Notice that $H_1 = 3x$ and $H_2 = d + e + f$.





- Prove that $de + ef + fd = 3x^2$.
- Prove that $d^2 + e^2 + f^2 \geq de + ef + fd$.
- Prove that $H_2 \geq H_1$
- Prove that $\frac{[DEF]}{[D'E'F']} \leq \frac{1}{4}$, with equality if and only if $\triangle DEF$ is equilateral.

17. Recall that, given any $\triangle ABC$, the area of the largest outer equilateral triangle of $\triangle ABC$ is $2[ABC] + \frac{\sqrt{3}}{6}(a^2 + b^2 + c^2)$. Use this and the previous problem to prove the **Weitzenböck inequality**:

$$4\sqrt{3}[ABC] \leq a^2 + b^2 + c^2.$$

4. Three dimensions

The concepts of inner, outer, and pedal triangles extend in the obvious way to tetrahedra.

18. Given a tetrahedron, is there a largest outer regular tetrahedron or a smallest inner regular tetrahedron? How would you find them? Could you calculate their volumes for a given tetrahedron?

19. Given a tetrahedron, is there a point P such that the pedal tetrahedron of P (defined in the expected way) is regular? What conditions must the given tetrahedron satisfy for a regular pedal tetrahedron to exist? If there is a point P for which the pedal tetrahedron is regular, what is the locus of such points?

5. Miquel Points, Apollonian Circles, Isodynamic Points

20. Given $\triangle ABC$, use the first isodynamic point to construct the smallest inner equilateral triangle and to calculate its area.

21. Given $\triangle ABC$, prove that the Apollonian circles are concurrent at two points (which are known as the first and second isodynamic points).

22. Given $\triangle ABC$, prove that the locus of points P such that the pedal triangle of P is isosceles, with vertex on side BC, is the Apollonian circle of $\triangle ABC$ passing through A.

23 (Miquel point)**.** Given $\triangle ABC$ and points X, Y, and Z respectively on sides BC, CA, and AB, prove that the circles (AYZ), (XBZ), and (XYC) are concurrent.

24. Prove[2] that there is a unique spiral similarity sending any segment AB to any other segment CD, unless $ABCD$ is a parallelogram.

25. Given segments $\overline{AB}$ and $\overline{CD}$, with $P = \overleftrightarrow{AD} \cap \overleftrightarrow{BC}$. Let M be the other point of intersection of (PAB) and (PDC). Prove that M is both the center of the spiral similarity sending $\overline{AB}$ to $\overline{DC}$ and the center of the spiral similarity sending $\overline{AD}$ to $\overline{BC}$.

26 (Miquel's Theorem)**.** Given the complete quadrilateral $ABCD$, with $P = \overleftrightarrow{AD} \cap \overleftrightarrow{BC}$ and $Q = \overleftrightarrow{AB} \cap \overleftrightarrow{CD}$, prove that the circles (PAB), (PDC), (QAD), and (QBC) are concurrent. Also, prove that the point of concurrency M, known as the *Miquel* point, is the center of spiral similarities sending $\overline{AB}$ to $\overline{DC}$ and $\overline{AD}$ to $\overline{BC}$.

27 (Isogonal Conjugate)**.** Given point P and $\triangle ABC$, let P_A be the reflection of P over the angle bisector of $\angle A$, let P_B be the reflection of P over the angle bisector of $\angle B$, and let P_C be the reflection of P over the angle bisector of $\angle C$. Use Ceva's theorem to prove that the lines AP_A, BP_B, and CP_C are concurrent at a point P^*, the *isogonal conjugate* of P.

28. Prove that the isogonal conjugate of the (first) isodynamic point is the Fermat point.

[2]This problem and the two following are taken almost verbatim from Evan Chen's truly excellent book **Euclidean Geometry in Mathematical Olympiads**.