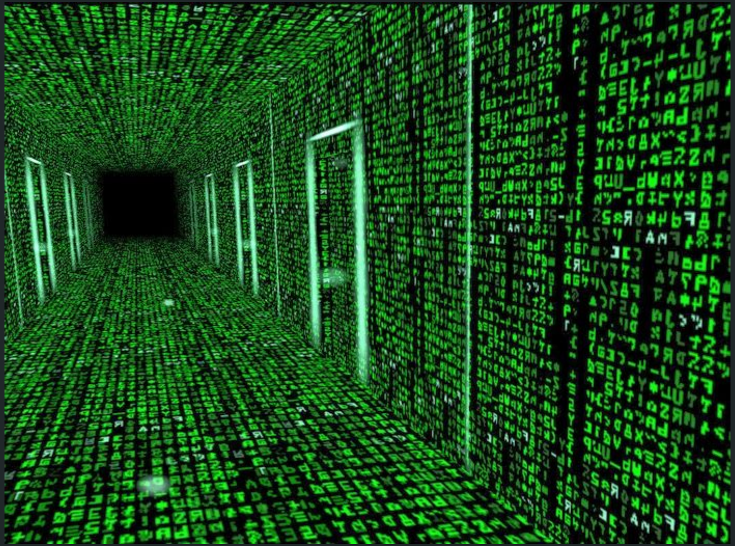
Matrices



Ellen Kulinsky

Amusement Parks

At an amusement park, each adult ticket costs $10 and each children's ticket costs $5. At the end of one day, the amusement park as sold $200 worth of tickets. You also know that in total 30 tickets were sold. How many adult tickets and how many children tickets were sold?

Money equation:

$10a + 5c = 200$

Num. of tickets equation:

$a + c = 30$

Substitution!

$a = 30 - c$

$10(30 - c) + 5c = 200$

$300 - 10c + 5c = 200$

$100 = 5c$

$c = 20$

$a = 10$

But that wasn't bad... Time to level up.

Get rid of y!

$$\left\{\begin{array}{l} 18x + 6y = 96 \\ 5x + 3y = 36 \end{array}\right. \quad \xrightarrow{\times 2} \quad \left.\begin{array}{l} 18x + 6y = 96 \\ 10x + 6y = 72 \end{array}\right\}$$

Subtract:

$$8x = 24 \longrightarrow x = 3 \xrightarrow{\text{Plug back in}}$$

$$5\times3 + 3y = 36$$

$$15 + 3y = 36$$

$$3y = 21$$

$$y = 7$$

What about now?

$$\left\{\begin{array}{l} 9x + 6y + z = 96 \\ 3x + \quad + 12z = 22 \\ x + 5y + 2z = 17 \end{array}\right.$$

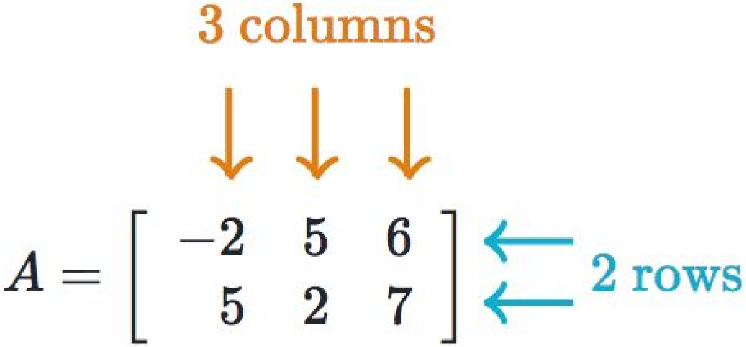
Substitution or Elimination…

is there another way?

YES! It’s called a matrix.

Matrix:

- a rectangular arrangement of numbers into rows and columns
- very useful way to represent information and work with data
- often used in computers



$$A = \begin{bmatrix} -2 & 5 & 6 \\ 5 & 2 & 7 \end{bmatrix}$$

Dimensions: m by n matrix (rows by columns) (m × n)

How does this have to do with systems of equations?

$$\begin{aligned} 10a + 5c &= 200 \\ a + c &= 30 \end{aligned} \quad \left(\begin{array}{cc|c} 10 & 5 & 200 \\ 1 & 1 & 30 \end{array}\right)$$

$$\begin{aligned} 18x + 6y &= 96 \\ 5x + 3y &= 36 \end{aligned} \quad \left(\begin{array}{cc|c} 18 & 6 & 96 \\ 5 & 3 & 36 \end{array}\right)$$

Question: Dimensions?

Answer: 2 by 3

Question: How does this relate to original equations?

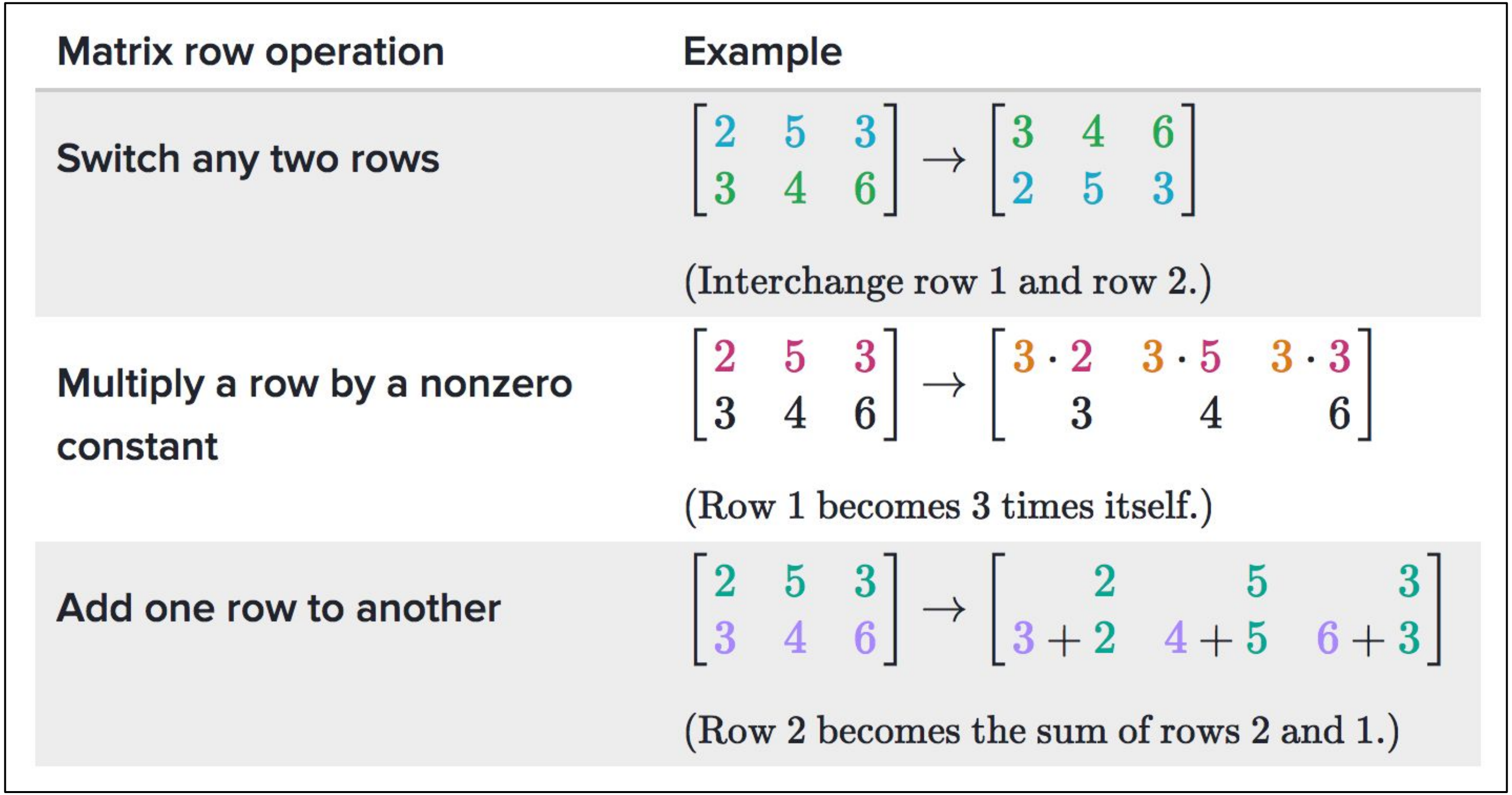
Answer:
# of rows = # of equations
# of columns = 1 + # of variables

Question: Why plus 1?

Answer: One column represents no variables

Row Operations

three basic operations that can be performed on a matrix without changing the solution set of the linear system it represents



Matrix row operation	Example
Switch any two rows	$\begin{bmatrix} 2 & 5 & 3 \\ 3 & 4 & 6 \end{bmatrix} \rightarrow \begin{bmatrix} 3 & 4 & 6 \\ 2 & 5 & 3 \end{bmatrix}$ (Interchange row 1 and row 2.)
Multiply a row by a nonzero constant	$\begin{bmatrix} 2 & 5 & 3 \\ 3 & 4 & 6 \end{bmatrix} \rightarrow \begin{bmatrix} 3 \cdot 2 & 3 \cdot 5 & 3 \cdot 3 \\ 3 & 4 & 6 \end{bmatrix}$ (Row 1 becomes 3 times itself.)
Add one row to another	$\begin{bmatrix} 2 & 5 & 3 \\ 3 & 4 & 6 \end{bmatrix} \rightarrow \begin{bmatrix} 2 & 5 & 3 \\ 3+2 & 4+5 & 6+3 \end{bmatrix}$ (Row 2 becomes the sum of rows 2 and 1.)

Row Operations

2x + 5y = 3
3x + 4y = 6

Row Operation	Equations
Switch any two rows	3x + 4y = 6 2x + 5y = 3 (Interchange row 1 and row 2)
Multiply a row by a nonzero constant	(3•2)x + (3•5)y = (3•3) 3x + 4y = 6 (Row 1 becomes 3 times itself)
Add one row to another	2x + 5y = 3 5x + 9y = 9 (Row 2 becomes the sum of rows 2 and 1)

ech·e·lon

Origin

late 18th century (sense 2 of the noun): from French *échelon*, from *échelle* ‘ladder,’ from Latin *scala* .

Reduced Row Echelon Form

A **pivot** is the first nonzero entry in a row.

In **Reduced Row Echelon Form:**

- every pivot is a one
- all other entries besides the pivot in pivot columns are zeros
- every following row has a pivot that is further right.

$$\boxed{\begin{bmatrix} 0 & 1 & 2 & 7 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}}$$

$$\begin{bmatrix} 5 & -2 & 1 \\ 0 & 3 & -1 \\ 2 & 0 & 7 \end{bmatrix}$$

$$\begin{bmatrix} 0 & 2 & 0 \\ 0 & 5 & 7 \\ 1 & 2 & 9 \end{bmatrix}$$

$$\boxed{\begin{bmatrix} 1 & 0 & 3 & -1 \\ 0 & 1 & -4 & 2 \\ 0 & 0 & 0 & 0 \end{bmatrix}}$$

$$\begin{bmatrix} 1 & 2 & 3 \\ 0 & -4 & 1 \\ 0 & 3 & -1 \end{bmatrix}$$

$$\boxed{\begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}}$$

$$\boxed{\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 2 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}}$$

$$\begin{bmatrix} 1 & 2 & 3 & 4 \\ -1 & 2 & 0 & 5 \\ 0 & 4 & -2 & 6 \\ 2 & 4 & 0 & 6 \end{bmatrix}$$

$$\boxed{\begin{bmatrix} 1 & -5 & 1 & 4 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}}$$

Review Time

What is a matrix?

- a rectangular arrangement of numbers into rows and columns
- very useful way to represent information and work with data
- often used in computers

What are the dimensions of a matrix?

$$\begin{array}{rccc} & \text{3 columns} & & \\ & \downarrow & \downarrow & \downarrow \\ A = \Bigg[& -2 & 5 & 6 \\ & 5 & 2 & 7 \end{array} \Bigg] \begin{array}{l} \leftarrow \\ \leftarrow \end{array} \text{2 rows}$$

- m by n matrix (m × n)
- rows by columns

Rewrite this system of equations as a coefficient matrix.

$$\left.\begin{aligned} 11x + 5y &= 99 \\ 17x + 14y &= 32 \end{aligned}\right\} \left(\begin{array}{cc|c} 11 & 5 & 99 \\ 17 & 14 & 32 \end{array}\right)$$

What are the three row operations?

Matrix row operation	Example

Review Time

What is a pivot?

- first nonzero entry in a row

What is reduced row echelon form?

- every pivot is a one
- all other entries in pivot column, except pivot, are zeros
- every following pivot is strictly further right

What are the steps to simplifying a matrix into reduced row echelon form?

1) Top left: 1.
2) Make all entries below: 0.
3) Make second entry, second row: 1.
4) All numbers, not pivot in column, turn into 0s.
5) Repeat until totally in R.R.E.F.

How to solve a system of equations using a matrix:

1) Rewrite as an “augmented” matrix.
2) Simplify into reduced row echelon form using row operations.

Your turn!

$$\left.\begin{aligned} 51x + 25y &= 101 \\ x + 34y &= 69 \\ 4x + 18y &= 40 \end{aligned}\right\} \left(\begin{array}{cc|c} 51 & 25 & 101 \\ 1 & 34 & 69 \\ 4 & 18 & 40 \end{array}\right)$$

Your turn!

$$\left.\begin{array}{r} 5x + 4y + 13z = 230 \\ x + 3y + 5z = 34 \\ 7x + 20z = 95 \end{array}\right\} \left.\begin{array}{r} 5x + 4y + 13z = 230 \\ 1x + 3y + 5z = 34 \\ 7x + 0y + 20z = 95 \end{array}\right\} \left(\begin{array}{ccc|c} 5 & 4 & 13 & 230 \\ 1 & 3 & 5 & 34 \\ 7 & 0 & 20 & 95 \end{array}\right)$$

Your turn!

$$\left.\begin{aligned} 5x - 4y + 13z &= 230 \\ -x + 3y + 5z &= 34 \\ 7x - 20z &= -95 \end{aligned}\right\}$$

$$\left.\begin{aligned} 5x + (-4)y + 13z &= 230 \\ (-1)x + 3y + 5z &= 34 \\ 7x + 0y + (-20)z &= -95 \end{aligned}\right\}$$

$$\left(\begin{array}{ccc|c} 5 & -4 & 13 & 230 \\ -1 & 3 & 5 & 34 \\ 7 & 0 & -20 & -95 \end{array}\right)$$

Your turn!

$$\begin{aligned} 3x - 2y &= 4 \\ x + 5z &= -3 \\ -4x - y + 3z &= 0 \end{aligned}$$

$$\begin{aligned} 3x + (-2)y + 0z &= 4 \\ 1x + 0y + 5z &= -3 \\ -4x + (-1)y + 3z &= 0 \end{aligned}$$

$$\begin{array}{cc}
\left[\begin{array}{cccc} 3 & -2 & 0 & 4 \\ 1 & 0 & 5 & -3 \\ -4 & -1 & 3 & 0 \end{array}\right] & \begin{array}{l} \leftarrow \text{Eq. 1} \\ \leftarrow \text{Eq. 2} \\ \leftarrow \text{Eq. 3} \end{array} \\
\begin{array}{cccc} \uparrow & \uparrow & \uparrow & \uparrow \\ x & y & \text{z} & \text{constants} \end{array} &
\end{array}$$

Practice!

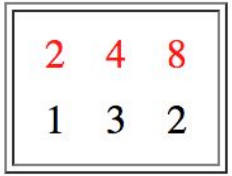
$$\begin{bmatrix} 1 & 5 & 7 \\ -2 & -7 & -5 \end{bmatrix} \xrightarrow{2R_1 + R_2} \begin{bmatrix} 1 & 5 & 7 \\ 0 & 3 & 9 \end{bmatrix} \xrightarrow{R_2 / 3} \begin{bmatrix} 1 & 5 & 7 \\ 0 & 1 & 3 \end{bmatrix} \xrightarrow{R_1 + (-5)R_2} \begin{bmatrix} 1 & 0 & -8 \\ 0 & 1 & 3 \end{bmatrix}$$

Steps:

1) Top left: 1.
2) Make second row start with: 0.
3) Make second entry, second row: 1.
4) All numbers, not pivot in column, turn into 0s.
5) Repeat until totally in R.R.E.F.

Your turn!

1) Top left: 1.
2) Make all entries below: 0.
3) Make second entry, second row: 1.
4) All numbers, not pivot in column, turn into 0s.
5) Repeat until totally in R.R.E.F.



2	4	8
1	3	2

Now let's combine it all!

To solve a system of equations using a matrix:

1) Rewrite as an "augmented" matrix.
2) Simplify into reduced row echelon form using row operations.

Amusement park:

$$\left.\begin{array}{r} 10a + 5c = 200 \\ a + c = 30 \end{array}\right\} \left(\begin{array}{cc|c} 10 & 5 & 200 \\ 1 & 1 & 30 \end{array}\right)$$

$$\begin{pmatrix} 10 & 5 & 200 \\ 1 & 1 & 30 \end{pmatrix} \xrightarrow{R_1 \leftrightarrow R_2} \begin{pmatrix} 1 & 1 & 30 \\ 10 & 5 & 200 \end{pmatrix} \xrightarrow{R_2 + (-10)R_1} \begin{pmatrix} 1 & 1 & 30 \\ 0 & -5 & -100 \end{pmatrix}$$

$$\xrightarrow{R_2 / (-5)} \begin{pmatrix} 1 & 1 & 30 \\ 0 & 1 & 20 \end{pmatrix} \xrightarrow[R_1 - R_2]{} \begin{pmatrix} 1 & 0 & 10 \\ 0 & 1 & 20 \end{pmatrix}$$

a = 10 tickets
c = 20 tickets

To solve a system of equations using a matrix:

1) Rewrite as an “augmented” matrix.
2) Simplify into RREF using row operations.

Your turn!

$$\left.\begin{aligned} a + 2b + 3c &= 9 \\ 2a - b + c &= 8 \\ 3a - c &= 3 \end{aligned}\right\} \quad \left[\begin{array}{ccc|c} 1 & 2 & 3 & 9 \\ 2 & -1 & 1 & 8 \\ 3 & 0 & -1 & 3 \end{array}\right]$$

1) Top left: 1.
2) Make all entries below: 0.
3) Make second entry, second row: 1.
4) All numbers, not pivot in column, turn into 0s.
5) Repeat until totally in R.R.E.F.

$$\left[\begin{array}{ccc|c} 1 & 2 & 3 & 9 \\ 0 & -5 & -5 & -10 \\ 0 & -6 & -10 & -24 \end{array}\right] \begin{array}{l} (\text{Row } 1) \\ (\text{Row } 2-2\cdot\text{Row } 1) \\ (\text{Row } 3-3\cdot\text{Row } 1) \end{array}$$

$$\left[\begin{array}{ccc|c} 1 & 2 & 3 & 9 \\ 0 & 1 & 1 & 2 \\ 0 & -6 & -10 & -24 \end{array}\right] \begin{array}{l} (\text{Row } 1) \\ (-1/5\cdot\text{Row } 2) \\ (\text{Row } 3) \end{array}$$

$$\left[\begin{array}{ccc|c} 1 & 2 & 3 & 9 \\ 0 & 1 & 1 & 2 \\ 0 & 0 & -4 & -12 \end{array}\right] \begin{array}{l} (\text{Row } 1) \\ (\text{Row } 2) \\ (\text{Row } 3+6\cdot\text{Row } 2) \end{array}$$

$$\left[\begin{array}{ccc|c} 1 & 2 & 3 & 9 \\ 0 & 1 & 1 & 2 \\ 0 & 0 & 1 & 3 \end{array}\right] \begin{array}{l} (\text{Row } 1) \\ (\text{Row } 2) \\ (-1/4\cdot\text{Row } 3) \end{array}$$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WORK


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GOOD
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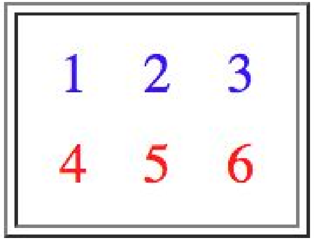
HW:

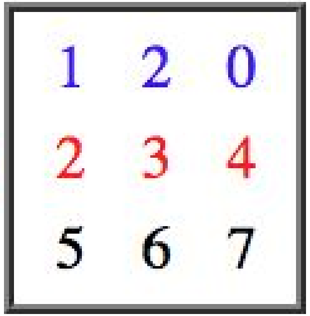
Reduce into reduced row echelon form:

1)



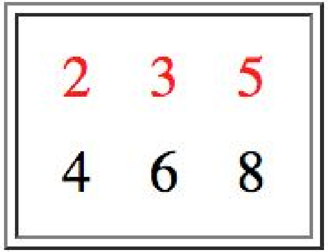
$$\begin{matrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{matrix}$$

2)



$$\begin{matrix} 1 & 2 & 0 \\ 2 & 3 & 4 \\ 5 & 6 & 7 \end{matrix}$$

*3)

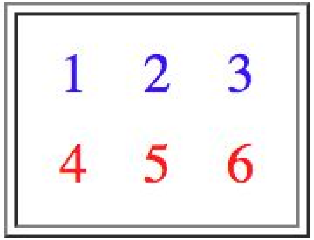


$$\begin{matrix} 2 & 3 & 5 \\ 4 & 6 & 8 \end{matrix}$$

HW:

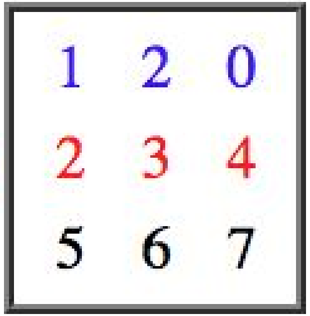
Reduce into reduced row echelon form:

1)



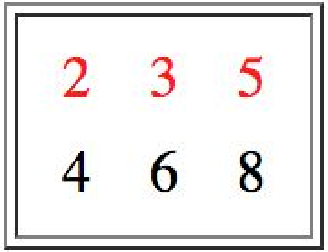
$$\begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix}$$

2)



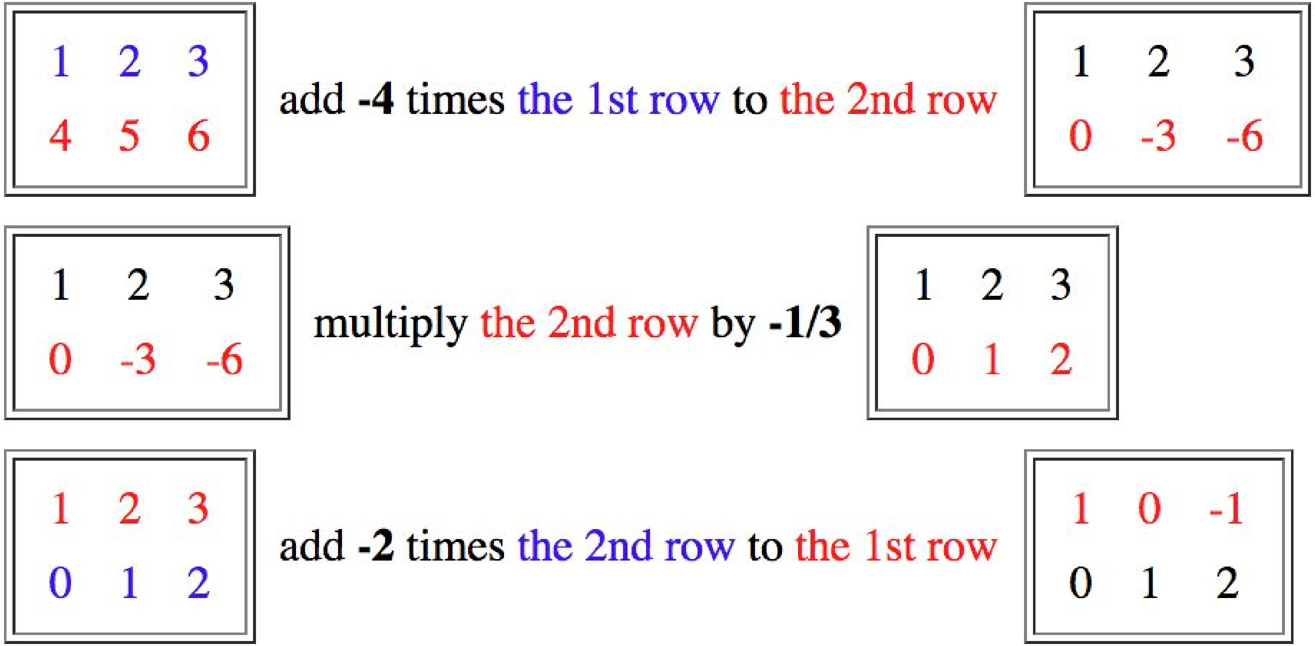
$$\begin{bmatrix} 1 & 2 & 0 \\ 2 & 3 & 4 \\ 5 & 6 & 7 \end{bmatrix}$$

*3)



$$\begin{bmatrix} 2 & 3 & 5 \\ 4 & 6 & 8 \end{bmatrix}$$

HW 1 Solution:

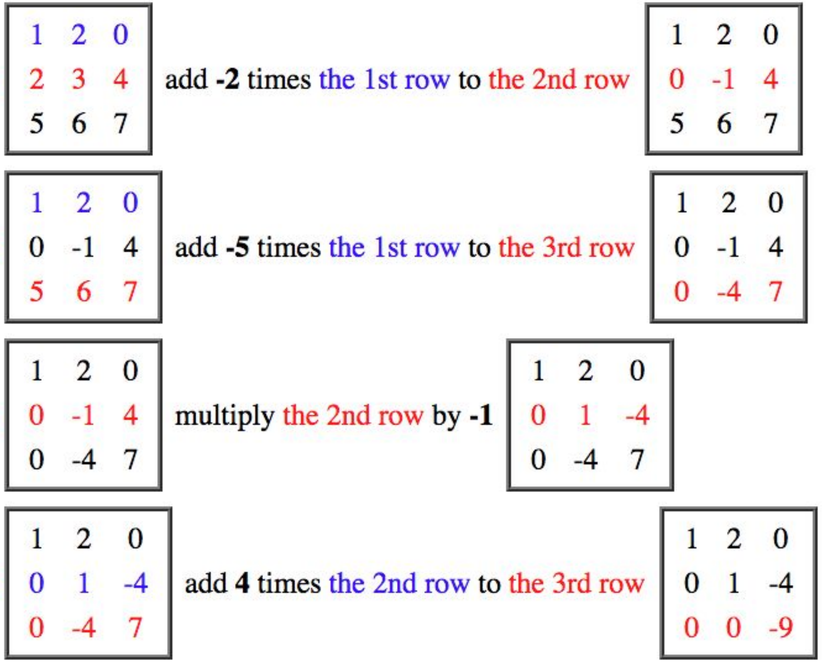


$\begin{matrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{matrix}$ add **-4** times the 1st row to the 2nd row $\begin{matrix} 1 & 2 & 3 \\ 0 & -3 & -6 \end{matrix}$

$\begin{matrix} 1 & 2 & 3 \\ 0 & -3 & -6 \end{matrix}$ multiply the 2nd row by **-1/3** $\begin{matrix} 1 & 2 & 3 \\ 0 & 1 & 2 \end{matrix}$

$\begin{matrix} 1 & 2 & 3 \\ 0 & 1 & 2 \end{matrix}$ add **-2** times the 2nd row to the 1st row $\begin{matrix} 1 & 0 & -1 \\ 0 & 1 & 2 \end{matrix}$

HW 2 Solution:

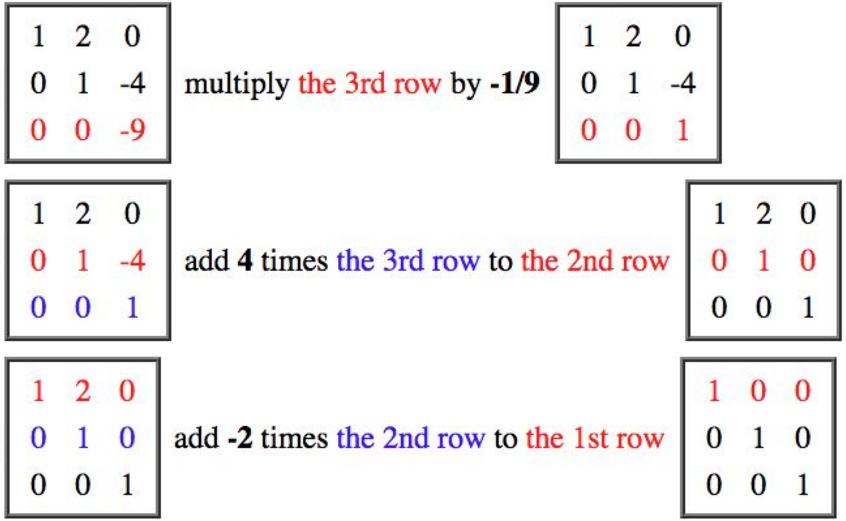


$\begin{matrix}1 & 2 & 0\\ 2 & 3 & 4\\ 5 & 6 & 7\end{matrix}$ add **-2** times the 1st row to the 2nd row $\begin{matrix}1 & 2 & 0\\ 0 & -1 & 4\\ 5 & 6 & 7\end{matrix}$

$\begin{matrix}1 & 2 & 0\\ 0 & -1 & 4\\ 5 & 6 & 7\end{matrix}$ add **-5** times the 1st row to the 3rd row $\begin{matrix}1 & 2 & 0\\ 0 & -1 & 4\\ 0 & -4 & 7\end{matrix}$

$\begin{matrix}1 & 2 & 0\\ 0 & -1 & 4\\ 0 & -4 & 7\end{matrix}$ multiply the 2nd row by **-1** $\begin{matrix}1 & 2 & 0\\ 0 & 1 & -4\\ 0 & -4 & 7\end{matrix}$

$\begin{matrix}1 & 2 & 0\\ 0 & 1 & -4\\ 0 & -4 & 7\end{matrix}$ add **4** times the 2nd row to the 3rd row $\begin{matrix}1 & 2 & 0\\ 0 & 1 & -4\\ 0 & 0 & -9\end{matrix}$

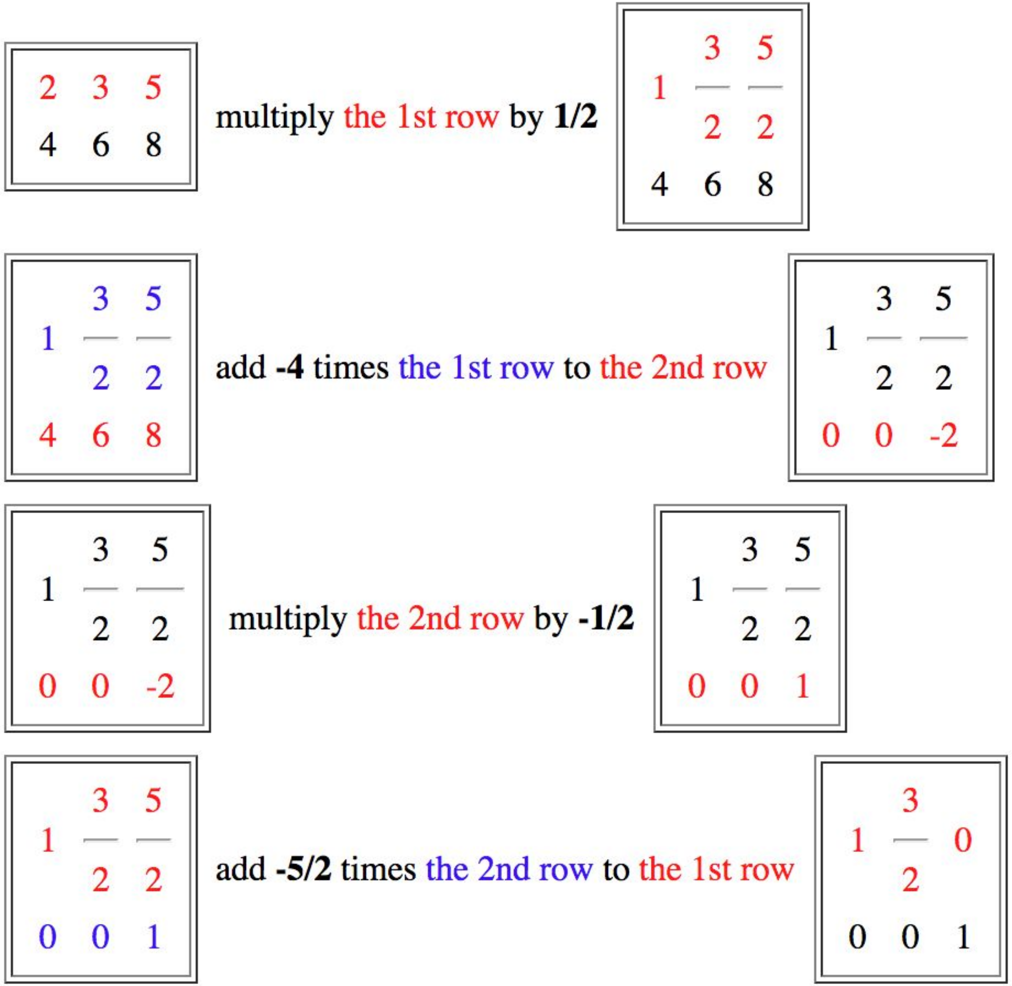


$\begin{matrix}1 & 2 & 0\\ 0 & 1 & -4\\ 0 & 0 & -9\end{matrix}$ multiply the 3rd row by **-1/9** $\begin{matrix}1 & 2 & 0\\ 0 & 1 & -4\\ 0 & 0 & 1\end{matrix}$

$\begin{matrix}1 & 2 & 0\\ 0 & 1 & -4\\ 0 & 0 & 1\end{matrix}$ add **4** times the 3rd row to the 2nd row $\begin{matrix}1 & 2 & 0\\ 0 & 1 & 0\\ 0 & 0 & 1\end{matrix}$

$\begin{matrix}1 & 2 & 0\\ 0 & 1 & 0\\ 0 & 0 & 1\end{matrix}$ add **-2** times the 2nd row to the 1st row $\begin{matrix}1 & 0 & 0\\ 0 & 1 & 0\\ 0 & 0 & 1\end{matrix}$

HW *3 Solution:



$$\begin{bmatrix}2&3&5\\4&6&8\end{bmatrix}$$

multiply the 1st row by **1/2**

$$\begin{bmatrix}1&\frac{3}{2}&\frac{5}{2}\\4&6&8\end{bmatrix}$$

$$\begin{bmatrix}1&\frac{3}{2}&\frac{5}{2}\\4&6&8\end{bmatrix}$$

add **-4** times the 1st row to the 2nd row

$$\begin{bmatrix}1&\frac{3}{2}&\frac{5}{2}\\0&0&-2\end{bmatrix}$$

$$\begin{bmatrix}1&\frac{3}{2}&\frac{5}{2}\\0&0&-2\end{bmatrix}$$

multiply the 2nd row by **-1/2**

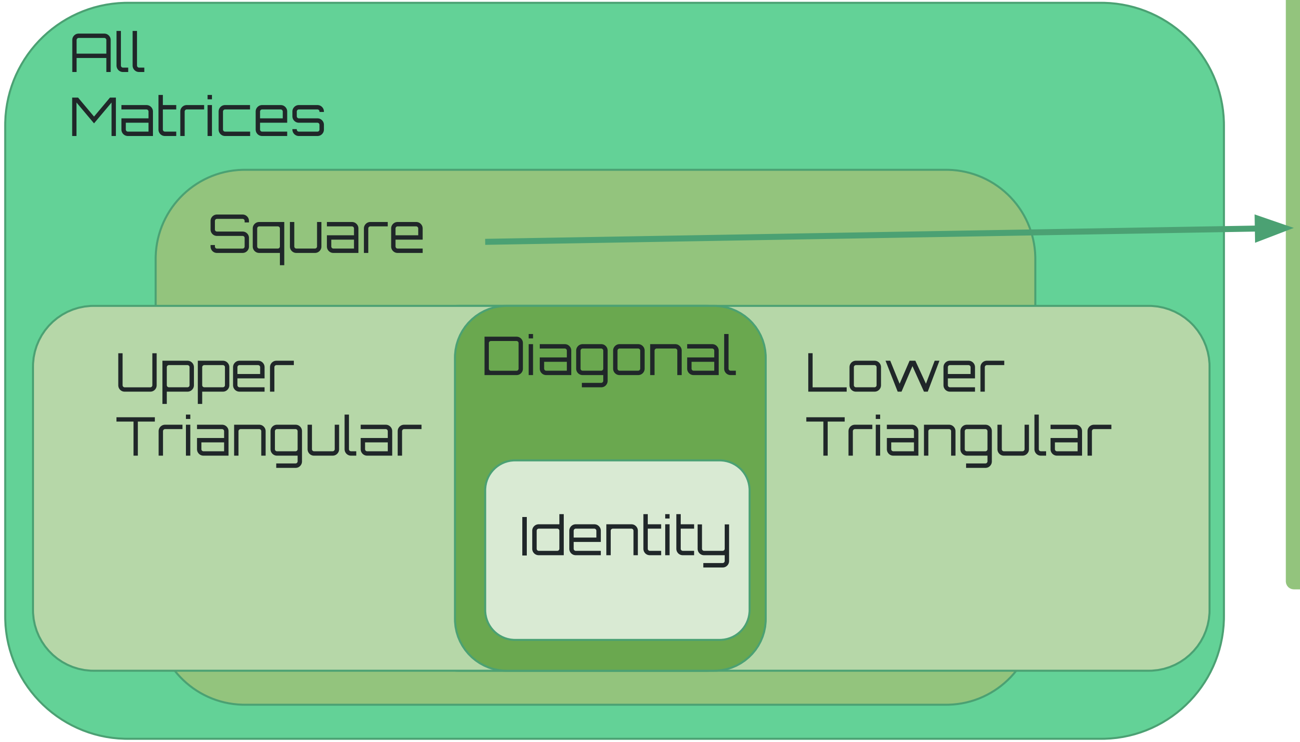
$$\begin{bmatrix}1&\frac{3}{2}&\frac{5}{2}\\0&0&1\end{bmatrix}$$

$$\begin{bmatrix}1&\frac{3}{2}&\frac{5}{2}\\0&0&1\end{bmatrix}$$

add **-5/2** times the 2nd row to the 1st row

$$\begin{bmatrix}1&\frac{3}{2}&0\\0&0&1\end{bmatrix}$$

Special Matrices



All Matrices

Square

Upper Triangular

Diagonal

Lower Triangular

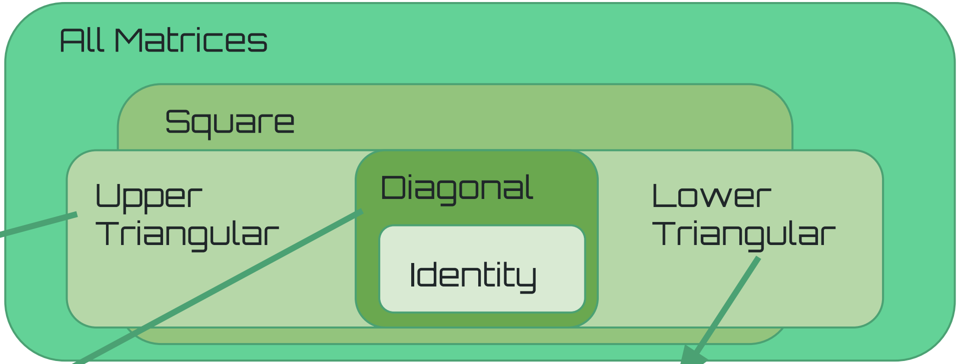
Identity

Square

Dimensions: n by n (same number of rows and columns)

$$\begin{pmatrix} 10 & 20 \\ 30 & 40 \end{pmatrix}$$

Special Matrices



All Matrices

Square

Upper Triangular

Diagonal

Identity

Lower Triangular

Upper Triangular

All entries below the diagonal are zeros.

$$\begin{pmatrix} 10 & 20 \\ 0 & 40 \end{pmatrix}$$

Diagonal

All entries except on the diagonal are zero

$$\begin{pmatrix} 10 & 0 \\ 0 & 40 \end{pmatrix}$$

Lower Triangular

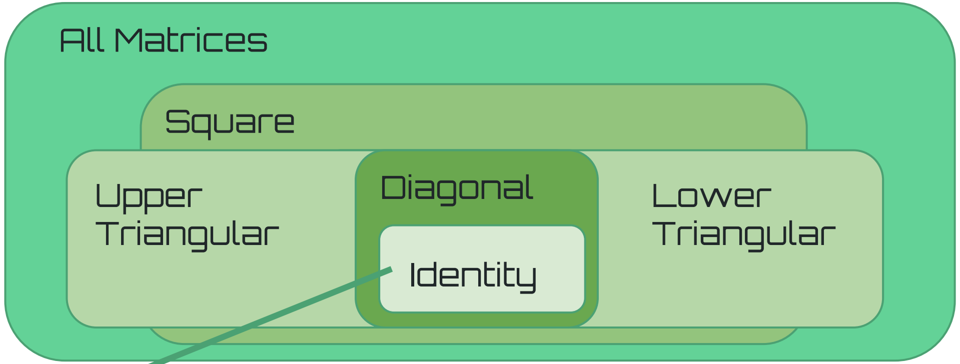
All entries above the diagonal are zeros.

$$\begin{pmatrix} 10 & 0 \\ 20 & 40 \end{pmatrix}$$

<u>Question</u>: RREF?

<u>Answer</u>: Upper Triangular

Special Matrices



Identity

All entries are zero, except 1s on the diagonal.

$$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

Question: Why is the identity special? And why is it called the identity matrix?

Answer:
If you multiply any matrix by the identity of the appropriate size, you will get back the same (an identical) matrix.

What operations can we do with matrices?

1) Matrix Addition (and Subtraction)
2) Scalar Multiplication (and Division)
3) Matrix Multiplication
4) Transpose
5) Determinant
6) Inverse

Matrix Addition

- Impossible to add matrices of different dimensions
- Matrices are added together by adding the corresponding elements

$$\begin{bmatrix} 0 & 1 & 2 \\ 9 & 8 & 7 \end{bmatrix} + \begin{bmatrix} 6 & 5 & 4 \\ 3 & 4 & 5 \end{bmatrix} =$$

Solve for x and y in the matrix below.

$$\begin{bmatrix} -3 & x \\ 2y & 0 \end{bmatrix} + \begin{bmatrix} 4 & 6 \\ -3 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 7 \\ -5 & 1 \end{bmatrix}$$

$x = 1$

$y = 3$

Matrix Subtraction

$$\begin{bmatrix} -1 & 2 & 0 \\ 0 & 3 & 6 \end{bmatrix} - \begin{bmatrix} 0 & -4 & 3 \\ 9 & -4 & -3 \end{bmatrix}$$

Solve for Matrix B.

$$B - \begin{bmatrix} 1 & 6 \\ 19 & 3 \end{bmatrix} = \begin{bmatrix} 4 & 2 \\ 8 & 1 \end{bmatrix}$$

$$B = \begin{bmatrix} 5 & 8 \\ 27 & 4 \end{bmatrix}$$

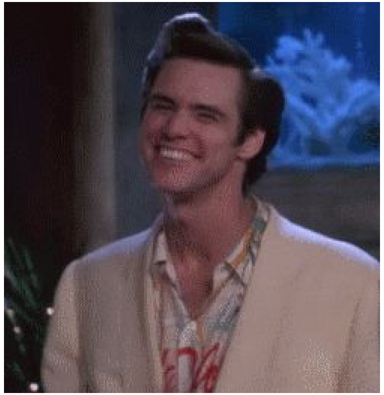
Scalar multiplication

- Multiplying a matrix by a scalar (number) results in every entry scaled by that number

Find 2A (multiply matrix A by the scalar 2).

$$A = \begin{bmatrix} 10 & 6 \\ 4 & 3 \end{bmatrix}$$

$$2A = 2 \cdot \begin{bmatrix} 10 & 6 \\ 4 & 3 \end{bmatrix} = \begin{bmatrix} 2 \cdot 10 & 2 \cdot 6 \\ 2 \cdot 4 & 2 \cdot 3 \end{bmatrix} = \begin{bmatrix} 20 & 12 \\ 8 & 6 \end{bmatrix}$$



Practice!

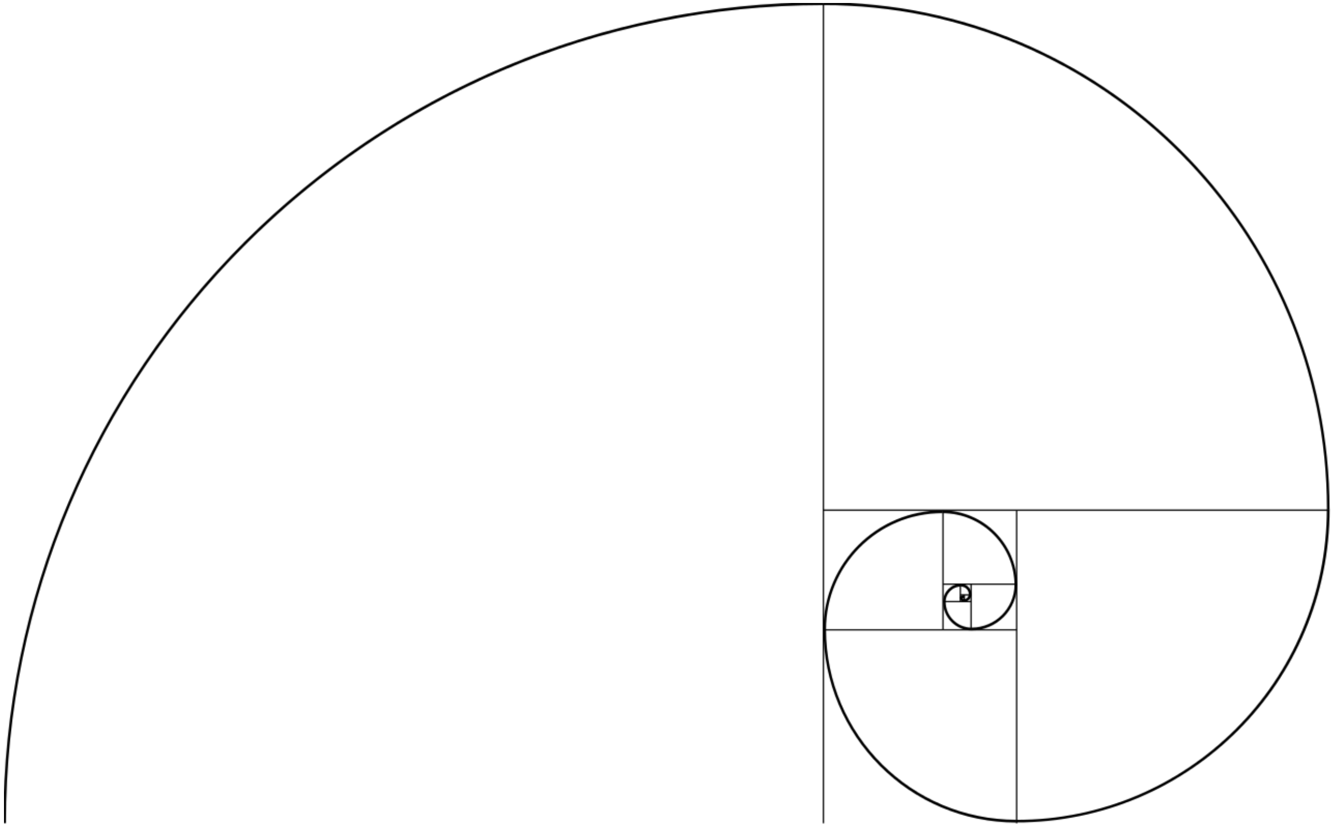
1) $$\begin{matrix}2&3&5\\4&6&8\end{matrix} + \begin{matrix}1&2&3\\4&5&6\end{matrix}$$

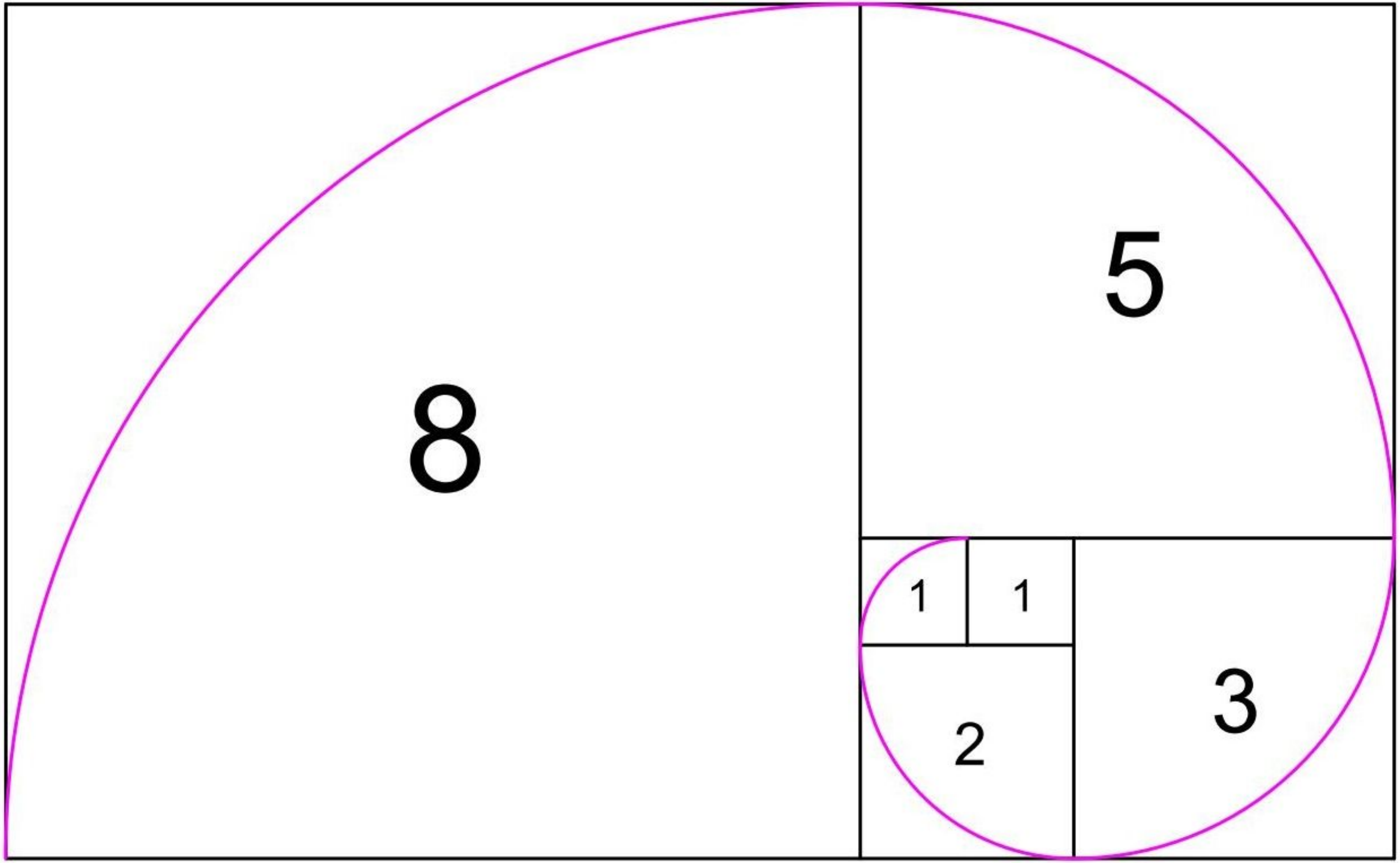
$$\begin{pmatrix}3&5&8\\8&11&14\end{pmatrix}$$

2) $$\begin{matrix}2&3&5\\4&6&8\end{matrix} - \begin{matrix}1&2&3\\4&5&6\end{matrix}$$

$$\begin{pmatrix}1&1&2\\0&1&2\end{pmatrix}$$

3) $$3 \times \begin{matrix}2&3&5\\4&6&8\end{matrix} = \begin{pmatrix}6&9&15\\12&18&24\end{pmatrix}$$

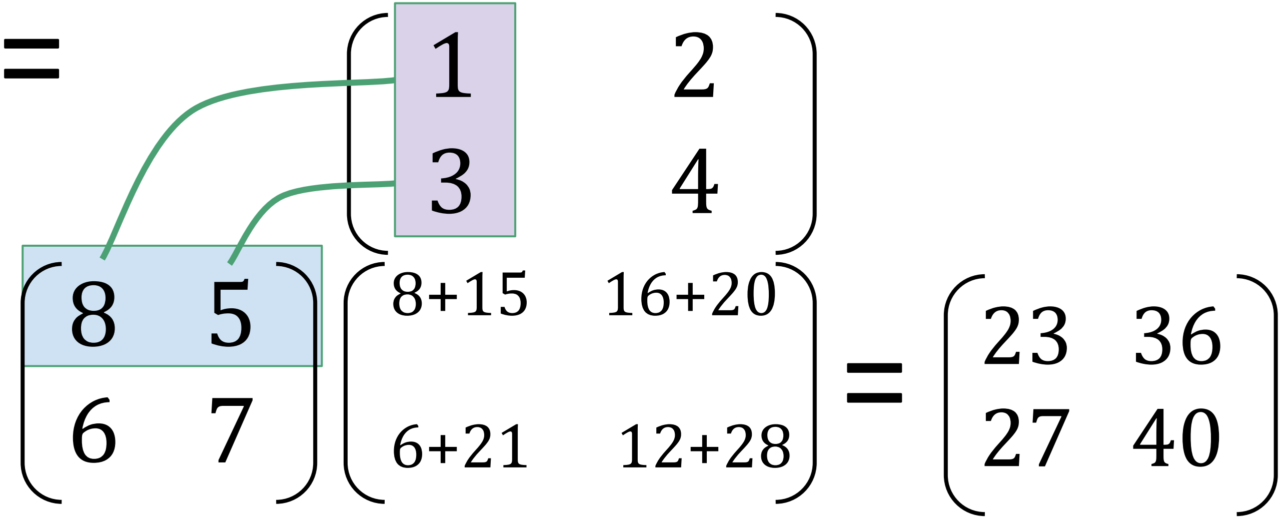



8
5
1
1
2
3

Matrix Multiplication

Hang on to your seats, this might get a little weird.

$$\begin{pmatrix}8&5\\6&7\end{pmatrix}\times\begin{pmatrix}1&2\\3&4\end{pmatrix}=$$



$$\begin{pmatrix}1&2\\3&4\end{pmatrix}$$

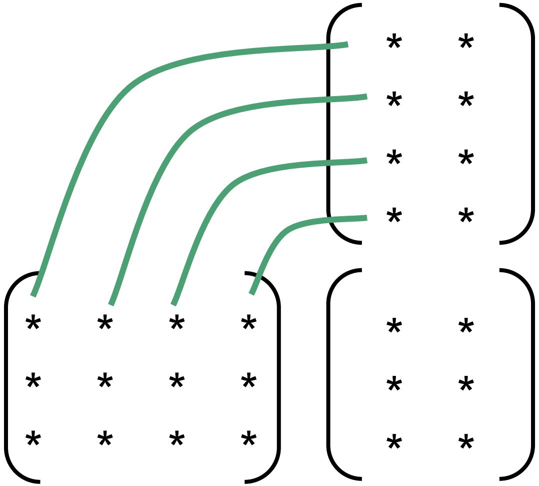
$$\begin{pmatrix}8&5\\6&7\end{pmatrix}\begin{pmatrix}8+15&16+20\\6+21&12+28\end{pmatrix}=\begin{pmatrix}23&36\\27&40\end{pmatrix}$$

Matrix Multiplication (skip if reviewed already)

Matrix A: 3 by 4

Matrix B: 4 by 2

A x B = C



Question: How many rows does result C have?

Answer: 3

Question: How many columns does result C have?

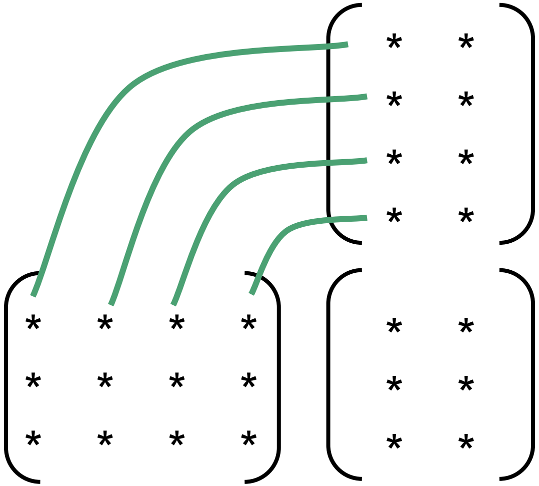
Answer: 2

Matrix Multiplication

Matrix A: a by b

A x B = C

Matrix B: c by d



<u>Question</u>: How many rows does result C have?

<u>Answer</u>: a

<u>Question</u>: How many columns does result C have?

<u>Answer</u>: d

<u>Question</u>: What do we know about the dimensions of A and B?

<u>Answer</u>: b=c
[The number of columns in A are equal to the number of rows in B]

Practice!!!

$$\begin{pmatrix}2 & 3\\ 1 & 0\end{pmatrix} \times \begin{pmatrix}1 & 2\\ 3 & 4\end{pmatrix} =$$

$$\times \quad \begin{pmatrix}1 & 2\\ 3 & 4\end{pmatrix}$$

$$\begin{pmatrix}2 & 3\\ 1 & 0\end{pmatrix}\begin{pmatrix}2\cdot 1 + 3\cdot 3 & 2\cdot 2 + 3\cdot 4\\ 1\cdot 1 + 0\cdot 3 & 1\cdot 2 + 0\cdot 4\end{pmatrix} = \begin{pmatrix}11 & 16\\ 1 & 2\end{pmatrix}$$

$$\begin{pmatrix}2 & 2\\ 4 & 1\\ 0 & 6\end{pmatrix} \times \begin{pmatrix}1 & -2 & 5\\ 0 & 4 & 3\end{pmatrix} = \begin{pmatrix}2 & 4 & 16\\ 4 & -4 & 23\\ 0 & 24 & 18\end{pmatrix}$$

Transpose

<u>Transpose</u>: A matrix that is obtained from flipping the original over its diagonal

Hint: Imagine placing a mirror on the diagonal

$$\begin{pmatrix} 10 & 20 & 30 \\ 40 & 50 & 60 \end{pmatrix} \xrightarrow{\text{Transpose}} \begin{pmatrix} 10 & 40 \\ 20 & 50 \\ 30 & 60 \end{pmatrix}$$

What happens if you transpose a matrix transpose?

Original Matrix

Practice!!!

$$\begin{pmatrix} 2 & 3 \\ 1 & 0 \end{pmatrix} \xrightarrow{\text{Transpose}} \begin{pmatrix} 2 & 1 \\ 3 & 0 \end{pmatrix}$$

$$\begin{pmatrix} 2 & 2 \\ 4 & 1 \\ 0 & 6 \end{pmatrix} \xrightarrow{\text{Transpose}} \begin{pmatrix} 2 & 4 & 0 \\ 2 & 1 & 6 \end{pmatrix}$$

$$\begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix} \xrightarrow{\text{Transpose}} \begin{pmatrix} 1 & 3 \\ 2 & 4 \end{pmatrix}$$

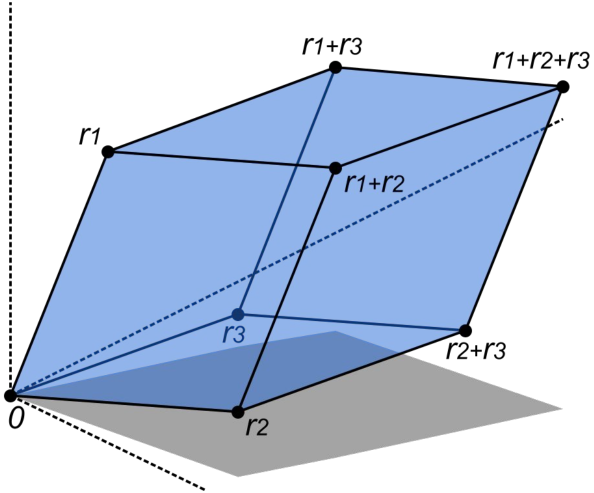
$$\begin{pmatrix} 1 & -2 & 5 \\ 0 & 4 & 3 \end{pmatrix} \xrightarrow{\text{Transpose}} \begin{pmatrix} 1 & 0 \\ -2 & 4 \\ 5 & 3 \end{pmatrix}$$

Determinant

Determinant: A number obtained from a *square* matrix, by following a certain algorithm

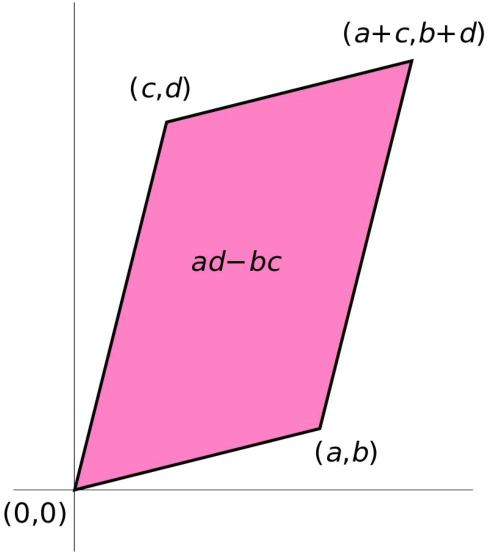
2 × 2 matrix determinant:

$$|A| = \begin{vmatrix} a & b \\ c & d \end{vmatrix} = ad - bc.$$



Geometric Interpretation:

- *The area of the parallelogram is the absolute value of the determinant of the matrix formed by the vectors representing the parallelogram's sides.*



Determinant (cont.)

2×2 matrix determinant:

$$|A| = \begin{vmatrix} a & b \\ c & d \end{vmatrix} = ad - bc.$$

<u>Notation:</u> Straight lines around a matrix (looks like an absolute value)

Example:

$$\begin{pmatrix} 10 & 20 \\ 12 & 40 \end{pmatrix} \rightarrow$$

10×40 - 20×12 =
400 - 240 = 160

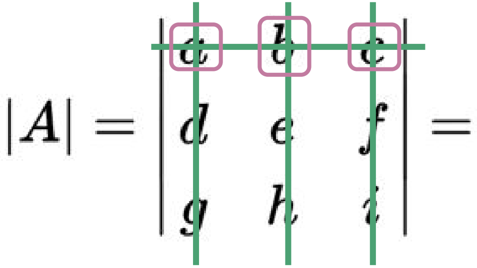
Practice:

$$\begin{pmatrix} 3 & 5 \\ 4 & 6 \end{pmatrix} \rightarrow$$

3×6 - 5×4 =
18 - 20 = -2

Determinant (cont.)

3 × 3 matrix determinant:



$$|A| = \begin{vmatrix} \boxed{a} & \boxed{b} & \boxed{c} \\ d & e & f \\ g & h & i \end{vmatrix} =$$

$$= a(ei - fh) - b(di - fg) + c(dh - eg)$$

$$= aei - afh - bdi + bfg + cdh - ceg$$

Fun fact: you can expand along any column or row

Expand along rows/columns with the most zeroes

Determinant (cont.)

How to know if positive or negative term:

$$\begin{pmatrix} + & - & + \\ - & + & - \\ + & - & + \end{pmatrix}$$

Row Number + Column Number

➔ even: positive term

➔ odd: negative term

Determinant (cont.)

$$\begin{pmatrix} 2 & 0 & 0 \\ 4 & 5 & 0 \\ 7 & 9 & 4 \end{pmatrix} \rightarrow 2\times5\times4 = 40$$

$$\begin{pmatrix} 2 & 1 & 0 \\ 4 & 5 & 0 \\ 7 & 9 & 4 \end{pmatrix} \rightarrow 2\times(5\times4 - 0) - 1\times(4\times4-0) = 40 - 16 = 24$$

$$\begin{pmatrix} 2 & 0 & 1 \\ 4 & 5 & 1 \\ 7 & 0 & 4 \end{pmatrix} \rightarrow 5\times(2\times4 - 1\times7) = 5\times1 = 5$$

BONUS:

$$\begin{pmatrix} 2 & 0 & 0 & 1 \\ 4 & 5 & 0 & 3 \\ 7 & 9 & 4 & 2 \\ 3 & 0 & 0 & 8 \end{pmatrix}$$

$$4 \times \begin{vmatrix} 2 & 0 & 1 \\ 4 & 5 & 3 \\ 3 & 0 & 8 \end{vmatrix} =$$

$$4 \times \left(5\times \begin{vmatrix} 2 & 1 \\ 3 & 8 \end{vmatrix} \right) =$$

$$4 \times (5\times(2\times8 - 3\times1)) =$$

$$4 \times 5 \times 13$$

Rules of Determinants

- det(identity) = 1
  - In fact, the determinant of any matrix with zeroes below or above the diagonal is just the product of the diagonal entries. Try to explain this.
- $\det(c \times A) = c^n \times \det(A)$ where n is the dimension of the square matrix. Try to explain this.
- For two square matrices of equal size, A and B:
  - $\det(AB) = \det(A) \times \det(B)$
- $\det(A^T) = \det(A)$

Inverse

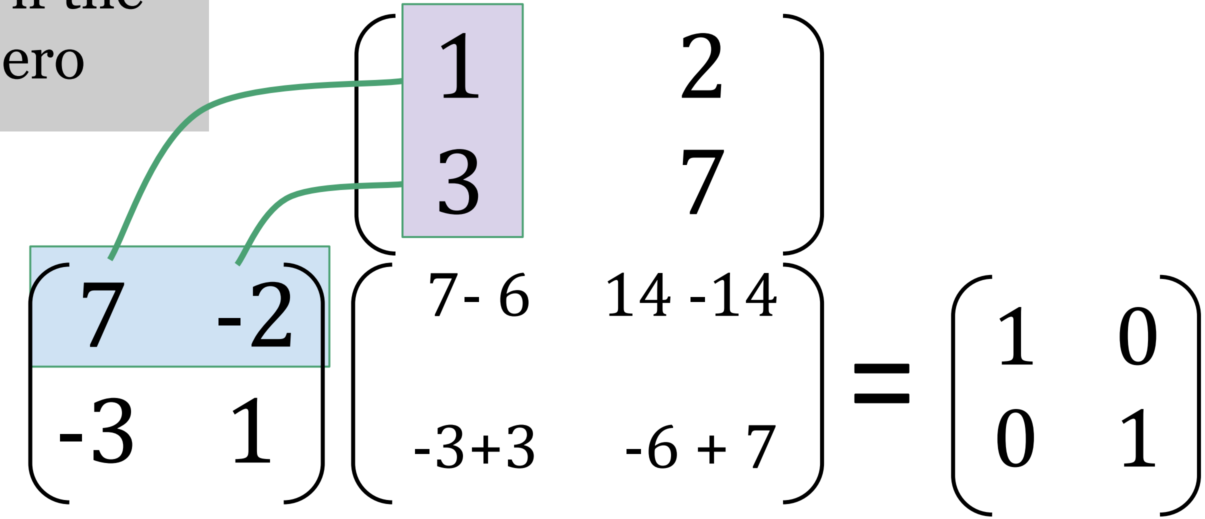
Similar to inverse of number: reciprocal

Inverse: A matrix when multiplied with the original matrix returns the identity; invertible if and only if the determinant is non-zero

8 → ⅛

8 × ⅛ = 1

Anything times 1 is itself!

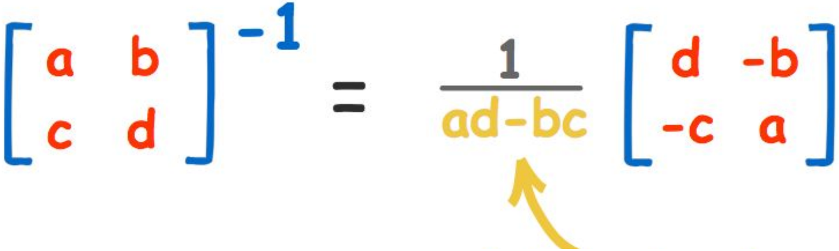


$$\begin{pmatrix} 1 & 2 \\ 3 & 7 \end{pmatrix}$$

$$\begin{pmatrix} 7 & -2 \\ -3 & 1 \end{pmatrix} \begin{pmatrix} 7-6 & 14-14 \\ -3+3 & -6+7 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

Inverse

2 × 2 matrix inverse:



$$\begin{bmatrix} a & b \\ c & d \end{bmatrix}^{-1} = \frac{1}{ad-bc}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$$

How do we find inverses for bigger matrices?

Steps:

1) Create an augmented matrix with the invertible square on one side and the identity of appropriate size on the other.
2) Reduce to RREF.
3) The new right side is the inverse.

Inverse

Example: $\left[\begin{array}{cc|cc} 1 & 2 & 1 & 0 \\ 3 & 4 & 0 & 1 \end{array}\right] \rightarrow$

What are the steps to simplifying a matrix into reduced row echelon form?

1) Top left: 1.
2) Make all entries below: 0.
3) Make second entry, second row: 1.
4) All numbers, not pivot in column, turn into 0s.
5) Repeat until totally in R.R.E.F.

Inverse

Why must the determinant be non-zero?

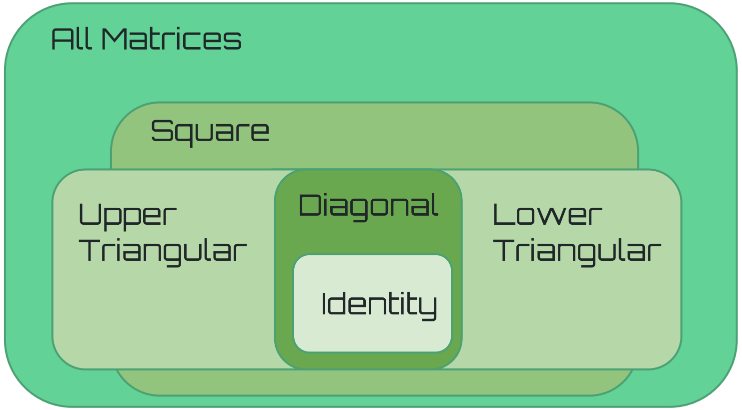
$\det(AB)=\det(A) \times \det(B)$

$AA^{-1} = I$

$\det(AA^{-1}) = \det(A) \times \det(A^{-1}) = \det(I) = 1$

$\det(A) = 1/\det(A^{-1})$

Review!



Square

Dimensions: n by n (same number of rows and columns)

Lower Triangular

All entries above the diagonal are zeros.

Upper Triangular

All entries below the diagonal are zeros.

Diagonal

All entries except on the diagonal are zero.

What operations can we do with matrices?

1) Matrix Addition (and Subtraction)
2) Scalar Multiplication (and Division)
3) Matrix Multiplication
4) Transpose
5) Determinant
6) Inverse

$$\begin{pmatrix}8&5\\6&7\end{pmatrix}+\begin{pmatrix}1&2\\3&4\end{pmatrix}=\begin{pmatrix}9&7\\9&11\end{pmatrix}$$

$$\begin{pmatrix}8&5\\6&7\end{pmatrix}-\begin{pmatrix}1&2\\3&4\end{pmatrix}=\begin{pmatrix}7&3\\3&3\end{pmatrix}$$

$$4\times\begin{pmatrix}8&5\\6&7\end{pmatrix}=\begin{pmatrix}32&20\\24&28\end{pmatrix}$$

Identity

All entries are zero, except 1s on the diagonal.

If you multiply any matrix by the identity of the appropriate size, you will get back the same (an identical) matrix.

Review!

What operations can we do with matrices?

1) Matrix Addition (and Subtraction)
2) Scalar Multiplication (and Division)
3) Matrix Multiplication
4) Transpose
5) Determinant
6) Inverse

$$\begin{pmatrix}2&3\\7&6\end{pmatrix} \times \begin{pmatrix}8&5\\4&1\end{pmatrix} =$$

$$\begin{pmatrix}8&5\\4&1\end{pmatrix}$$

$$\begin{pmatrix}2&3\\7&6\end{pmatrix}\begin{pmatrix}16+12&10+3\\56+24&35+6\end{pmatrix} = \begin{pmatrix}28&13\\80&41\end{pmatrix}$$

Determinant:

$$\begin{pmatrix}2&3\\7&6\end{pmatrix} \rightarrow 2\times6 - 7\times3 = 12 - 21 = -9$$

$$\begin{pmatrix}2&0&0\\4&5&0\\7&9&4\end{pmatrix} \rightarrow 2\times5\times4 = 40$$

$$\begin{pmatrix}3&5\\7&9\\1&4\end{pmatrix} \rightarrow$$

Trick question, not a square!

$$\begin{pmatrix}2&1&3\\4&5&1\\7&9&4\end{pmatrix} \rightarrow 2\times(5\times4 - 1\times9) -1\times(4\times4 - 1\times7) +3\times(4\times9 - 5\times7) = 2\times11 - 1\times9 + 3\times1 = 22 - 9 + 3 = 16$$

Transpose:

$$\begin{pmatrix}2&3\\7&6\end{pmatrix} \rightarrow \begin{pmatrix}2&7\\3&6\end{pmatrix}$$

$$\begin{pmatrix}3&5\\7&9\\1&4\end{pmatrix} \rightarrow \begin{pmatrix}3&7&1\\5&9&4\end{pmatrix}$$

Review!

What operations can we do with matrices?

1) Matrix Addition (and Subtraction)
2) Scalar Multiplication (and Division)
3) Matrix Multiplication
4) Transpose
5) Determinant
6) Inverse

Determinant:

$$|A| = \begin{vmatrix} a & b & c \\ d & e & f \\ g & h & i \end{vmatrix} = a\begin{vmatrix} e & f \\ h & i \end{vmatrix} - b\begin{vmatrix} d & f \\ g & i \end{vmatrix} + c\begin{vmatrix} d & e \\ g & h \end{vmatrix}$$

$$= a(ei - fh) - b(di - fg) + c(dh - eg)$$

$$= aei - afh - bdi + bfg + cdh - ceg$$

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Review!

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2 × 2 matrix inverse:

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Sources

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