A number system is a system for expressing numbers using a specific set of digits. The first system that the ancient people came up with required to put one stroke for each object. This was very inconvenient as to denote a collection of twenty objects you would need twenty strokes. The next idea was to divide objects into small groups. We guess that because (most) people have five fingers on each of the two hands, groups of five and groups of ten were the most popular. When we talk about *base $\boldsymbol{n}$ system* we mean that we count units, groups of $\boldsymbol{n}$, groups of $\boldsymbol{n^2}$ and so on.

It is very easy to see what we mean by this when $\boldsymbol{n}$ is **ten**. We are accustomed to writing numbers in *base ten*, or *decimal system*, using the symbols 0, 1, 2, 3, 4, 5, 6, 7, 8, and 9. For example, seventy-nine is written as 79, and means seven tens and nine units. However numbers can be written in any base $\boldsymbol{n}$ system.

For example, if we use base **two** instead of base ten, then we only have two symbols: 0 and 1. So to write number "two" itself in base two, we will write **10**: this will mean one "two" and zero units. Number "three" has one "two" and one unit, so $\mathbf{3_{10} = 11_2}$ (the index below the number denotes which base we are using).

Continue and get further numbers:

$\mathbf{4_{10}}$ is the square of the base two, so $\mathbf{4_{10} = 100_2}$, $\mathbf{5_{10} = 101_2}$, $\mathbf{6_{10} = 110_2}$, $\mathbf{7_{10} = 111_2}$ and $\mathbf{8_{10} = 1000_2}$! Just as, in base ten, the columns represent powers of 10 and have 'place value' $\mathbf{1, 10, 10^2, 10^3}$ etc. (reading from right to left), so in base two, the columns represent powers of **2**: $\mathbf{1, 2, 2^2, 2^3}$ etc. Please remember, the symbols **2** and **3** are not present in the base two system, so when we talk about base two system and write $\mathbf{2^3}$ we simply use it to shorten the phrase "two to the power of three". The numbers above are written in the usual, base ten, system.

How about writing seventy-nine in base **2**? $\mathbf{79_{10}} = \boxed{}_{\mathbf{2}}$.

There are other bases, too. In Computer Science, bases two, eight and sixteen are used a lot. Can you write seventy-nine in base eight?

$\mathbf{79_{10}} = \boxed{}_{\mathbf{8}}$.

Base sixteen is interesting because we need 16 different symbols: mathematicians agreed to denote these as $\boldsymbol{0, 1, 2, 3, 4, 5, 6, 7, 8, 9, A, B, C, D, E, F}$. This means that number ten is written in base **16** as a one-digit number $\boldsymbol{A}$. What about one hundred? As $\mathbf{96 = 16 \times 6}$, we have $\mathbf{100 = 16 \times 6 + 4}$, so $\mathbf{100_{10} = 64_{16}}$. And one hundred and twenty-five? $\mathbf{125 - 16 \times 7 = 13}$, so $\boldsymbol{125_{10} = 7D_{16}}$. Now write seventy-nine in the base **16** system:

$\mathbf{79_{10}} = \boxed{}_{\mathbf{16}}$.

Fractions

What about fractions? To write numbers between 0 and 1, we use *negative powers* of the base (like decimal expansion in base ten). For example, in base 2 we use halves, quarters, eighths, sixteenths etc instead of the tenths, hundredths, thousandths etc. which we use in base ten. Note that the units are on the "zeroeth" place, so we multiply the units by the base in power $\mathbf{0}$ – this is a multiplication by $\mathbf{1}$.

So if we write $\mathbf{11.11_2}$ (in base two) this denotes $\mathbf{2^1 + 2^0 + 2^{-1} + 2^{-2}}$. The equivalent in base ten is $\mathbf{2 + 1 + \frac{1}{2} + \frac{1}{4}}$, that is $\mathbf{3.75_{10}}$ (in base $\mathbf{10}$). Of course $\mathbf{3.75_{10} = \frac{15}{4}_{10} = \frac{1111_2}{100_2} = \frac{1111}{100}_2}$. We proved that $\mathbf{11.11_2 = \frac{1111}{100}_2}$.

Please note that when we write a fraction in the form $\boldsymbol{\frac{m}{n}}$ (simple fraction) in a particular base, we mean that the digits used for $\boldsymbol{m}$ and $\boldsymbol{n}$ in that base determine what numbers we divide.

Let us now take $\mathbf{\frac{1}{10}_3}$ (in base $\mathbf{3}$). This is one over three (because $\mathbf{10_3 = 3_{10}}$), so it's a third, and then it's $\mathbf{3^{-1}}$ which is written as $\mathbf{0.1_3}$ (base $\mathbf{3}$). So, $\mathbf{\frac{1}{10}_3 = 0.1_3}$ – not so surprising after all! You can notice that unlike decimal fractions, a third in base $\mathbf{3}$ is a very pleasant fraction. Can you guess in what other base systems will a third be a finite fraction?

Maybe this example will help a little: Let us write $\mathbf{\frac{1}{8}_{10}}$ in bases $\mathbf{4}$ and $\mathbf{16}$. This fraction is twice as big as $\mathbf{\frac{1}{16}_{10}}$, so

$\mathbf{\frac{1}{8}_{10} = 0.02_4}$ and $\mathbf{\frac{1}{8}_{10} = 0.2_{16}}$.

1 Warm up questions:

Question 1. *Write numbers ten, fourteen and fifty in the following number systems:*

Base 2	*Base 3*	*Base 7*	*Base 16*
$\mathbf{10_{10}} =$	$\mathbf{10_{10}} =$	$\mathbf{10_{10}} =$	$\mathbf{10_{10}} =$
$\mathbf{14_{10}} =$	$\mathbf{14_{10}} =$	$\mathbf{14_{10}} =$	$\mathbf{14_{10}} =$
$\mathbf{50_{10}} =$	$\mathbf{50_{10}} =$	$\mathbf{50_{10}} =$	$\mathbf{50_{10}} =$

Question 2. *Write fractions (in the form* $\mathbf{0.a_1a_2\ldots}$ *) in the following number systems:*

Base 2	*Base 3*	*Base 7* (not so easy)*	*Base 16*
$\mathbf{0.75_{10}} =$	$\mathbf{\frac{1}{27}}_{\mathbf{10}} =$	$\mathbf{\frac{1}{14}}_{\mathbf{10}} =$	$\mathbf{0.5_{10}} =$

Question 3. *Write these words in decimal (base* **10***) system:*

$\boldsymbol{A1D_{16}} =$	$\boldsymbol{5EA_{16}} =$	$\boldsymbol{CAB_{16}} =$	$\boldsymbol{1CE_{16}} =$

Question 4. *A* **1110** *year old boy James has just started Grade* **1001***. With only* **100** *books in his schoolbag, he was very happy that his new Maths teacher, Miss Numebase, gave him the puzzle to solve: how is that possible that her dog has* **100** *legs,* **10** *eyes and* **1** *tail? Can you help James? By the way, what can you say about his age and other strange things?*

Question 5. *In what number base system does the equality* $\mathbf{3 \times 4 = 10}$ *hold?*

Question 6. *Is there a number base system in which both the following two conditions are satisfied simultaneously:*

A) $\mathbf{3 + 4 = 10}$ *and* $\mathbf{3 \times 4 = 15}$*?*

B) $\mathbf{2 + 3 = 5}$ *and* $\mathbf{2 \times 3 = 11}$*?*

Question 7. *George wrote a sum in his Maths book with an ink pen but then spilled water on his book. As a result some of the digits have disappeared. This is what Miss Numebase read:*

$$\begin{array}{cccccc} & 2 & 3 & \square & 5 & \square \\ + & 1 & \square & 6 & 4 & 2 \\ \hline & 4 & 2 & 4 & 2 & 3 \end{array}$$

Find out in which number base system he was adding the numbers and restore the missing digits.

Question 8. *Miss Numebase claims that there are* **100** *children in the classroom,* **24** *of them are boys and* **32** *are girls. What number base system does she use?*

Question 9. *We call a number composite if it is neither prime nor* **1***. In other words, a number is composite if it can be written as a product of two numbers, each strictly bigger than* **1***.*

A) Show that $\mathbf{10201_n}$ *is composite in any base.*

B) Likewise show that* $\mathbf{10101_n}$ *is composite in any base.*

2 Divisibility tests:

Question 10. *Find a condition that allows one to determine the parity of a number (is it even or is it odd) by its record*

A) in the ternary (base $\mathbf{3}$*) number system;*

B) in the base $\boldsymbol{n}$ *number system. Will your answer depend on* $\boldsymbol{n}$*?*

Question 11. *Find a condition that allows one to determine*

A) Divisibility by $\boldsymbol{3^k}$ *of a number by its record in the ternary system;*

B) Divisibility by $\boldsymbol{n^k}$ *of a number by its record in the base* $\boldsymbol{n}$ *number system;*

C) Divisibility by $\mathbf{3}$ *of a number by its record in the base* $\mathbf{6}$ *number system;*

D) Divisibility by $\boldsymbol{d}$ *of a number by its record in the base* $\boldsymbol{n}$ *number system, where* $\boldsymbol{n}$ *is divisible by* $\boldsymbol{d}$.

Question 12. *Find a condition that allows one to determine*

A) Divisibility by $\mathbf{3}$ *of a number by its record in the base* $\mathbf{7}$ *system;*

B) Divisibility by $\mathbf{4}$ *of a number by its record in the base* $\mathbf{9}$ *number system;*

C) Divisibility by $\boldsymbol{d}$ *of a number by its record in the base* $\boldsymbol{n}$ *number system, where* $\boldsymbol{n-1}$ *is divisible by* $\boldsymbol{d}$*;*

D) Divisibility by $\mathbf{3}$ *of a number by its record in the binary (base* $\mathbf{2}$*) system (hint: think first what would be its analog in the decimal system?);*

E) Divisibility by $\boldsymbol{n+1}$ *of a number by its record in the base* $\boldsymbol{n}$ *system;*

F) Divisibility by $\mathbf{5}$ *of a number by its record on the base* $\mathbf{9}$ *system;*

G) Divisibility by $\boldsymbol{d}$ *of a number by its record in the base* $\boldsymbol{n}$ *number system, where* $\boldsymbol{n+1}$ *is divisible by* $\boldsymbol{d}$.

3 More advanced questions:

Question 13. *Alice multiplied some real number by* $\mathbf{10}$ *and got a prime number. Bob multiplied the same real number by* $\mathbf{13}$ *and ... the result was also a prime number!*

Find out, in which base $\boldsymbol{n}$ *system (the same) they were multiplying. Is such* $\boldsymbol{n}$ *unique? Why?*

Question 14. *What is the smallest number of weights necessary to be able to weigh any number of grams from 1 to 100 on the balance scales with 2 weighing dishes, if weights can only be placed on one of the dishes?*

Question 15. *What is the smallest number of weights necessary in order to be able to weigh any number of grams from 1 to 100 on the balance scales, if weights can be placed on both dishes?*

Question 16. *The Queen of Hearts thinks of three two-digit numbers:* $\boldsymbol{A}$, $\boldsymbol{B}$ *and* $\boldsymbol{C}$. *Alice should give the queen three numbers:* $\boldsymbol{x}$, $\boldsymbol{y}$ *and* $\boldsymbol{z}$ *after which the Queen will give Alice the sum* $\boldsymbol{Ax + By + Cz}$. *Alice must guess the numbers* $\boldsymbol{A}$, $\boldsymbol{B}$ *and* $\boldsymbol{C}$ *from her first attempt, otherwise her head will be chopped off. How can Alice be saved?*

Question 17. *A) Prove that from the set* $\boldsymbol{\{0, 1, 2, \cdots, 3^k - 1\}}$ *, it is possible to choose* $\boldsymbol{2^k}$ *numbers so that none of them is the arithmetic mean of the other two chosen numbers.*

B) Prove that from the set $\boldsymbol{\{0, 1, 2, \cdots, \frac{1}{2}(3^k - 1)\}}$, *it is possible to choose* $\boldsymbol{2^k}$ *numbers so that none of them is the arithmetic mean of the other two chosen numbers.*