A number system is a system for expressing numbers using a specific set of digits. The first system that the ancient people came up with required to put one stroke for each object. This was very inconvenient as to denote a collection of twenty objects you would need twenty strokes. The next idea was to divide objects into small groups. We guess that because (most) people have five fingers on each of the two hands, groups of five and groups of ten were the most popular. When we talk about *base $\boldsymbol{n}$ system* we mean that we count units, groups of $\boldsymbol{n}$, groups of $\boldsymbol{n^2}$ and so on.

It is very easy to see what we mean by this when $\boldsymbol{n}$ is **ten**. We are accustomed to writing numbers in *base ten*, or *decimal system*, using the symbols 0, 1, 2, 3, 4, 5, 6, 7, 8, and 9. For example, seventy-nine is written as 79, and means seven tens and nine units. However numbers can be written in any base $\boldsymbol{n}$ system.

For example, if we use base **two** instead of base ten, then we only have two symbols: 0 and 1. So to write number "two" itself in base two, we will write **10**: this will mean one "two" and zero units. Number "three" has one "two" and one unit, so $\mathbf{3_{10} = 11_2}$ (the index below the number denotes which base we are using).

Continue and get further numbers:

$\mathbf{4_{10}}$ is the square of the base two, so $\mathbf{4_{10} = 100_2}$, $\mathbf{5_{10} = 101_2}$, $\mathbf{6_{10} = 110_2}$, $\mathbf{7_{10} = 111_2}$ and $\mathbf{8_{10} = 1000_2}$! Just as, in base ten, the columns represent powers of 10 and have 'place value' $\mathbf{1, 10, 10^2, 10^3}$ etc. (reading from right to left), so in base two, the columns represent powers of **2**: $\mathbf{1, 2, 2^2, 2^3}$ etc. Please remember, the symbols **2** and **3** are not present in the base two system, so when we talk about base two system and write $\mathbf{2^3}$ we simply use it to shorten the phrase "two to the power of three". The numbers above are written in the usual, base ten, system.

How about writing seventy-nine in base **2**? $\mathbf{79_{10} = \boxed{1001111}_{\,2}}$.

There are other bases, too. In Computer Science, bases two, eight and sixteen are used a lot. Can you write seventy-nine in base eight?

$\mathbf{79_{10} = \boxed{117}_{\,8}}$.

Base sixteen is interesting because we need 16 different symbols: mathematicians agreed to denote these as $\boldsymbol{0, 1, 2, 3, 4, 5, 6, 7, 8, 9, A, B, C, D, E, F}$. This means that number ten is written in base **16** as a one-digit number $\boldsymbol{A}$. What about one hundred? As $\mathbf{96 = 16 \times 6}$, we have $\mathbf{100 = 16 \times 6 + 4}$, so $\mathbf{100_{10} = 64_{16}}$. And one hundred and twenty-five? $\mathbf{125 - 16 \times 7 = 13}$, so $\boldsymbol{125_{10} = 7D_{16}}$. Now write seventy-nine in the base **16** system:

$\boldsymbol{79_{10} = \boxed{4F}_{\,16}}$.

Fractions

What about fractions? To write numbers between 0 and 1, we use *negative powers* of the base (like decimal expansion in base ten). For example, in base 2 we use halves, quarters, eighths, sixteenths etc instead of the tenths, hundredths, thousandths etc. which we use in base ten. Note that the units are on the "zeroeth" place, so we multiply the units by the base in power $\mathbf{0}$ – this is a multiplication by $\mathbf{1}$.

So if we write $\mathbf{11.11_2}$ (in base two) this denotes $\mathbf{2^1+2^0+2^{-1}+2^{-2}}$. The equivalent in base ten is $\mathbf{2+1+\frac{1}{2}+\frac{1}{4}}$, that is $\mathbf{3.75_{10}}$ (in base $\mathbf{10}$). Of course $\mathbf{3.75_{10}=\frac{15}{4}_{10}=\frac{1111_2}{100_2}=\frac{1111}{100}_2}$. We proved that $\mathbf{11.11_2=\frac{1111}{100}_2}$.

Please note that when we write a fraction in the form $\boldsymbol{\frac{m}{n}}$ (simple fraction) in a particular base, we mean that the digits used for $\boldsymbol{m}$ and $\boldsymbol{n}$ in that base determine what numbers we divide.

Let us now take $\mathbf{\frac{1}{10}_3}$ (in base $\mathbf{3}$). This is one over three (because $\mathbf{10_3=3_{10}}$), so it's a third, and then it's $\mathbf{3^{-1}}$ which is written as $\mathbf{0.1_3}$ (base $\mathbf{3}$). So, $\mathbf{\frac{1}{10}_3=0.1_3}$ – not so surprising after all! You can notice that unlike decimal fractions, a third in base $\mathbf{3}$ is a very pleasant fraction. Can you guess in what other base systems will a third be a finite fraction?

Maybe this example will help a little: Let us write $\mathbf{\frac{1}{8}_{10}}$ in bases $\mathbf{4}$ and $\mathbf{16}$. This fraction is twice as big as $\mathbf{\frac{1}{16}_{10}}$, so

$\mathbf{\frac{1}{8}_{10}=0.02_4}$ and $\mathbf{\frac{1}{8}_{10}=0.2_{16}}$.

1 Warm up questions:

Question 1. *Write numbers ten, fourteen and fifty in the following number systems:*

Solution:

Base 2	Base 3	Base 7	Base 16
$\mathbf{10_{10}=1010}$	$\mathbf{10_{10}=101}$	$\mathbf{10_{10}=13}$	$\mathbf{10_{10}}=\boldsymbol{A}$
$\mathbf{14_{10}=1110}$	$\mathbf{14_{10}=112}$	$\mathbf{14_{10}=20}$	$\mathbf{14_{10}}=\boldsymbol{E}$
$\mathbf{50_{10}=110010}$	$\mathbf{50_{10}=1212}$	$\mathbf{50_{10}=101}$	$\mathbf{50_{10}=32}$

Question 2. *Write fractions (in the form* $\boldsymbol{0.a_1a_2\dots}$ *) in the following number systems:*

Solution:

Base 2	Base 3	Base 7* (not so easy)	Base 16
$\mathbf{0.75_{10} = 0.11}$	$\mathbf{\frac{1}{27}_{10} = 0.001}$	$\mathbf{\frac{1}{14}_{10} = 0.03333\dots}$	$\mathbf{0.5_{10} = 0.8}$

Question 3. *Write these words in decimal (base* **10***) system:*

Solution:

$\boldsymbol{A1D_{16} = 2589}$	$\boldsymbol{5EA_{16} = 1514}$	$\boldsymbol{CAB_{16} = 3243}$	$\boldsymbol{1CE_{16} = 462}$

Question 4. *A* **1110** *year old boy James has just started Grade* **1001***. With only* **100** *books in his schoolbag, he was very happy that his new Maths teacher, Miss Numebase, gave him the puzzle to solve: how is that possible that her dog has* **100** *legs,* **10** *eyes and* **1** *tail? Can you help James? By the way, what can you say about his age and other strange things?*

Solution:

We are talking about base **2** system, of course. James is **14** year old, and he has just started Grade **9**. He has **4** books in his schoolbag, and the dog has **4** legs, **2** eyes and **1** tail.

Question 5. *In what number base system does the equality* $\mathbf{3 \times 4 = 10}$ *hold?*

Solution:

In a number system with base $\boldsymbol{n}$, this number $\boldsymbol{n}$ will be written as **10**. Therefore $\boldsymbol{n}$ is twelve.

Question 6. *Is there a number base system in which both the following two conditions are satisfied simultaneously:*

A) $\mathbf{3+4=10}$ *and* $\mathbf{3\times 4=15}$*?*

B) $\mathbf{2+3=5}$ *and* $\mathbf{2\times 3=11}$*?*

Solution:

A) As before, we see that $\boldsymbol{n}$ is equal to seven. We then check that $\mathbf{3}$ times $\mathbf{4}$ is twelve which will be written as 15 in number system with base seven.
$\underline{\boldsymbol{n}=\mathbf{7}.}$

B) $\mathbf{2+3=5}$ means that $\boldsymbol{n}$ needs to be more than $\mathbf{5}$, so at least $\mathbf{6}$. However, then $\mathbf{2}$ times $\mathbf{3}$ is either $\mathbf{6}$ (if $\boldsymbol{n}>\mathbf{6}$) or $\mathbf{10}$ (if $\boldsymbol{n}=\mathbf{6}$). This is a contradiction.
<u>There is no such number base system.</u>

Question 7. *George wrote a sum in his Maths book with an ink pen but then spilled water on his book. As a result some of the digits have disappeared. This is what Miss Numebase read:*

$$\begin{array}{cccccc} & 2 & 3 & \square & 5 & \square \\ + & 1 & \square & 6 & 4 & 2 \\ \hline & 4 & 2 & 4 & 2 & 3 \end{array}$$

Find out in which number base system he was adding the numbers and restore the missing digits.

Solution:

Let $\boldsymbol{n}$ be the base for this number system, and put $\boldsymbol{a, b, c}$ in the squares:

$$\begin{array}{cccccc} & \mathbf{2} & \mathbf{3} & \boxed{\boldsymbol{b}} & \mathbf{5} & \boxed{\boldsymbol{a}} \\ + & \mathbf{1} & \boxed{\boldsymbol{c}} & \mathbf{6} & \mathbf{4} & \mathbf{2} \\ \hline & \mathbf{4} & \mathbf{2} & \mathbf{4} & \mathbf{2} & \mathbf{3} \end{array}$$

In the right-most sum $\boxed{\text{a}}$ $\mathbf{+\,2=\ldots 3}$ we must have $\boldsymbol{a}+\mathbf{2}=\mathbf{3}$, i.e. $\boldsymbol{a}=\mathbf{1}$. Otherwise, $\boldsymbol{a}+\mathbf{2}\geq \boldsymbol{n}+\mathbf{3}$ which implies $\boldsymbol{a}\geq \boldsymbol{n}+\mathbf{1}$, which means $\boldsymbol{a}$ is not a digit. Hence we put it here:

$$\begin{array}{cccccc} & \mathbf{2} & \mathbf{3} & \boxed{\boldsymbol{b}} & \mathbf{5} & \boxed{\mathbf{1}} \\ + & \mathbf{1} & \boxed{\boldsymbol{c}} & \mathbf{6} & \mathbf{4} & \mathbf{2} \\ \hline & \mathbf{4} & \mathbf{2} & \mathbf{4} & \mathbf{2} & \mathbf{3} \end{array}$$

Our next sum is: $\mathbf{5+4 = \ldots 2}$. This means that $\mathbf{5+4}$ is equal to one of the numbers $\mathbf{12, 22, 32, 42, \ldots}$, or, in other words $\boldsymbol{kn}+\mathbf{2}$ where $\boldsymbol{k} \geq \mathbf{1}$. Note that $\mathbf{9} = \boldsymbol{kn}+\mathbf{2}$ implies $\mathbf{7} = \boldsymbol{kn}$ which is possible only if $\boldsymbol{n} = \mathbf{7}$ or $\boldsymbol{n} = \mathbf{1}$. Of course $\boldsymbol{n} > \mathbf{1}$ as we are using digits $\mathbf{2, 3, 4, 5, 6}$, so we get $\boldsymbol{n} = \mathbf{7}$. Then indeed, $\mathbf{5+4 = 12}$, and we get to the next sum:
$\mathbf{1} + \boxed{\text{b}} + \mathbf{6} = \mathbf{\ldots 4}$. We can rewrite this as $\mathbf{7} + \boxed{\text{b}} = \mathbf{7}\boldsymbol{k} + \mathbf{4}$ (we already know the value for $\boldsymbol{n}$).

This would only be possible if $\boldsymbol{b} = \mathbf{4}$:

$$\begin{array}{cccccc} & \mathbf{2} & \mathbf{3} & \boxed{\mathbf{4}} & \mathbf{5} & \boxed{\mathbf{1}} \\ + & \mathbf{1} & \boxed{\boldsymbol{c}} & \mathbf{6} & \mathbf{4} & \mathbf{2} \\ \hline & \mathbf{4} & \mathbf{2} & \mathbf{4} & \mathbf{2} & \mathbf{3} \end{array}$$

Finally, the next sum becomes $\mathbf{1+3} + \boxed{\text{c}} = \mathbf{\ldots 2}$, which we rewrite as $\mathbf{4} + \boxed{\text{c}} = \mathbf{7}\boldsymbol{k} + \mathbf{2}$. From this we find $\boldsymbol{c} = \mathbf{5}$:

$$\text{In Base } \mathbf{7}\text{:} \quad \begin{array}{cccccc} & \mathbf{2} & \mathbf{3} & \boxed{\mathbf{4}} & \mathbf{5} & \boxed{\mathbf{1}} \\ + & \mathbf{1} & \boxed{\mathbf{5}} & \mathbf{6} & \mathbf{4} & \mathbf{2} \\ \hline & \mathbf{4} & \mathbf{2} & \mathbf{4} & \mathbf{2} & \mathbf{3} \end{array}$$

Question 8. *Miss Numebase claims that there are* **100** *children in the classroom,* **24** *of them are boys and* **32** *are girls. What number base system does she use?*

Solution:

Let $\boldsymbol{n}$ be the base of the system, then there are $\boldsymbol{n^2}$ children in the class. We also have:
$\mathbf{24} = \mathbf{2}\boldsymbol{n} + \mathbf{4}$, $\mathbf{32} = \mathbf{3}\boldsymbol{n} + \mathbf{2}$

So, we need to solve the following equation $(\mathbf{2}\boldsymbol{n}+\mathbf{4}) + (\mathbf{3}\boldsymbol{n}+\mathbf{2}) = \boldsymbol{n^2}$, or, equivalently,
$\mathbf{5}\boldsymbol{n} + \mathbf{6} = \boldsymbol{n^2} \qquad \Longleftrightarrow \qquad \boldsymbol{n^2} - \mathbf{5}\boldsymbol{n} - \mathbf{6} = \mathbf{0}$.

The solutions are $\boldsymbol{n_{1,2}} = \dfrac{\mathbf{5} \pm \sqrt{\mathbf{25+24}}}{\mathbf{2}} = \dfrac{\mathbf{5 \pm 7}}{\mathbf{2}}$ so $\boldsymbol{n} = \mathbf{6}$ or $\boldsymbol{n} = \mathbf{-1}$.

Of course $\boldsymbol{n}$ cannot be negative, so we get $\boldsymbol{n} = \mathbf{6}$.

Let us check: there are $\mathbf{100_6 = 36}$ children in the class, $\mathbf{24_6 = 16}$ of the are boys and $\mathbf{32_6 = 20}$ are girls.

Question 9. *We call a number composite if it is neither prime nor* $\mathbf{1}$*. In other words, a number is composite if it can be written as a product of two numbers, each strictly bigger than* $\mathbf{1}$*.*

A) Show that $\mathbf{10201}_n$ *is composite in any base.*

B) Likewise show that* $\mathbf{10101}_n$ *is composite in any base.*

Solution:

A) We assume here $n \geq 3$. Note that $10201_n = 1 + 2n^2 + n^4 = (n^2 + 1)^2$, a composite number.

Alternative solution: $10201_n = 101_n \times 101_n$.

Indeed, $101_n \times 100_n = 10100_n$, and adding 101_n does not cause any sums to be bigger than $2 < n$, so $101_n \times 101_n = 101_n \times 100_n + 101_n = 10100_n + 101_n = 10201_n$.

B*) We assume here $n \geq 2$. Note that $10101_n = 1 + n^2 + n^4 = (1 + 2n^2 + n^4) - n^2 = (n^2 + 1)^2 - n^2 = (n^2 - n + 1) \times (n^2 + n + 1)$.

Note that $n^2 - n = n \times (n - 1) \geq n$, so $n^2 - n + 1 \geq n + 1 \geq 3$. Therefore both $n^2 - n + 1$ and $n^2 + n + 1$ are strictly bigger than 1.

2 Divisibility tests:

Question 10. *Find a condition that allows one to determine the parity of a number (is it even or is it odd) by its record*

A) in the ternary (base $\mathbf{3}$*) number system;*

B) in the base n *number system. Will your answer depend on* n*?*

Solution:

A) The number is even if and only if the sum of all the digits is even. It is equivalent to the fact that there are even number of ones in its ternary record.

Indeed, if the number's ternary representation is $\overline{a_k a_{k-1} \dots a_1 a_0}_{\;3}$, then it is equal to $a_0 + 3a_1 + 3^2 a_2 + \cdots + 3^k a_k$. This has the same parity as $a_0 + a_1 + a_2 + \cdots + a_k$. Since 0s and 2s add up to an even number, we should only be concerned with counting 1s.

B) If <u>n is an odd number</u>, then the answer is the same as the first answer in the previous part: The number is even if and only if the sum of all the digits is even.

Indeed, we only replace $\boldsymbol{3^i}$ by $\boldsymbol{n^i}$ and notice that the parity of $\boldsymbol{a_i n^i}$ is the same as that of $\boldsymbol{a_i}$. So the whole number $\boldsymbol{a_0 + na_1 + n^2 a_2 + \cdots + n^k a_k}$ has the same parity as $\boldsymbol{a_0 + a_1 + a_2 + \cdots + a_k}$.

If $\boldsymbol{n}$ is an even number, then the parity of a number written in base $\boldsymbol{n}$ number system coincides with that of its last digit.

Indeed, if the number's representation is $\overline{\boldsymbol{a_k a_{k-1} \dots a_1 a_0}}_{\ \boldsymbol{n}}$, then it is equal to $\boldsymbol{a_0 + na_1 + n^2 a_2 + \cdots + n^k a_k}$. All terms which involve powers of $\boldsymbol{n}$ are even, hence do not influence the parity. Hence the number's parity is the same as that of $\boldsymbol{a_0}$.

Question 11. *Find a condition that allows one to determine*

A) Divisibility by $\mathbf{3^k}$ *of a number by its record in the ternary system;*

B) Divisibility by $\boldsymbol{n^k}$ *of a number by its record in the base* $\boldsymbol{n}$ *number system;*

C) Divisibility by $\mathbf{3}$ *of a number by its record in the base* $\mathbf{6}$ *number system;*

D) Divisibility by $\boldsymbol{d}$ *of a number by its record in the base* $\boldsymbol{n}$ *number system, where* $\boldsymbol{n}$ *is divisible by* $\boldsymbol{d}$*.*

Solution:

We notice that A) is a particular case of B), and C) is a particular case of D).

B) A number is divisible by $\boldsymbol{n^k}$ if and only if its representation in the base $\boldsymbol{n}$ number system ends in $\boldsymbol{k}$ zeros.

D) Let $\boldsymbol{d}$ be a divisor of $\boldsymbol{n}$. Then all powers of $\boldsymbol{n}$ are divisible by $\boldsymbol{d}$, so have no influence on divisibility of the number by $\boldsymbol{d}$ (compare with Question 10, B)). Hence the last digit of the base $\boldsymbol{n}$ representation of the number is divisible by $\boldsymbol{d}$ if and only if the number itself is divided by $\boldsymbol{d}$.

C) We now have $\boldsymbol{d = 3}$ and $\boldsymbol{n = 6}$. So the number is divisible by $\mathbf{3}$ if and only if its last digit in the base $\mathbf{6}$, is divisible by $\mathbf{3}$. For instance, $\mathbf{27}$ in base $\mathbf{6}$ would be $\mathbf{43_6}$ – it is divisible by $\mathbf{3}$ because its last digit is $\mathbf{3}$.

Question 12. *Find a condition that allows one to determine*

A) Divisibility by $\mathbf{3}$ *of a number by its record in the base* $\mathbf{7}$ *system;*

B) Divisibility by $\mathbf{4}$ *of a number by its record in the base* $\mathbf{9}$ *number system;*

C) Divisibility by $\boldsymbol{d}$ *of a number by its record in the base* $\boldsymbol{n}$ *number system, where* $\boldsymbol{n-1}$ *is divisible by* $\boldsymbol{d}$*;*

D) Divisibility by $\mathbf{3}$ *of a number by its record in the binary (base* $\mathbf{2}$*) system (hint: think first what would be its analog in the decimal system?);*

E) Divisibility by $n + 1$ of a number by its record in the base n system;

F) Divisibility by 5 of a number by its record on the base 9 system;

G) Divisibility by d of a number by its record in the base n number system, where $n+1$ is divisible by d.

Solution:

A) and B): note that $7 - 1$ is divisible by 3, and $9 - 1$ is divisible by 4. Therefore the answers to A) and B) will follow from C).

C) Let d be a divisor of $n - 1$, and the number has the following representation in the base n system: $\overline{a_k a_{k-1} \dots a_1 a_0}_{\ n}$, i.e. the number itself is equal to $a_0 + a_1 n + a_2 n^2 + \dots a_k n^k$. Notice that each of the powers of n has remainder 1 with respect to divisibility by $n - 1$, hence the same is true with respect to divisibility by d. Therefore, the remainder of the number with respect to divisibility by d is the same as $a_0 + a_1 \times 1 + a_2 \times 1 + \cdots + a_k \times 1 = a_0 + a_1 + a_2 + \dots a_k$ has. Hence the answer is:

The sum of the digits of a number in the base n system is divisible by d if and only if the number itself is divisible by d.

This allows us to write explicit rules in:

A) The sum of the digits of a number in the base 7 system is divisible by 3 if and only if the number itself is divisible by 3. For example, 36 is written as 51_7, and we can see that $5 + 1$ is divisible by 3.

B) The sum of the digits of a number in the base 9 system is divisible by 4 if and only if the number itself is divisible by 4. For example, 36 is written as 40_9, and we can see that $4 + 0$ is divisible by 4.

All of D), E) and F) are particular cases of G). Indeed, in D) we have $n + 1 = 2 + 1$ is divisible by $d = 3$, in E) we have $d = n + 1$, so of course $n + 1$ is divisible by d, and in F) we have $n + 1 = 9 + 1$ is divisible by 5.

G) Let again the number be written as $\overline{a_k a_{k-1} \dots a_1 a_0}_{\ n}$ in the base n system, so it is equal to $a_0 + a_1 n + a_2 n^2 + \dots a_k n^k$. We notice that for each *odd* power $2i + 1$ of n we have that $n^{2i+1} + 1$ is divisible by $n + 1$: use, for instance, the formula

$$n^{2i+1} + 1 = (n + 1)(n^{2i} - n^{2i-1} + n^{2i-2} \mp \cdots + 1).$$

For each *even* power $2i$ of n, however, we have that $n^{2i} - 1$ is divisible by $n + 1$: write $n^{2i} - 1 = (n^i - 1)(n^i + 1)$ and either decompose further $n^i - 1$, if i is even, or get that $n^i + 1$ is divisible by $n + 1$, if i is odd.

Therefore,

$$\left(a_0 + a_1 n + a_2 n^2 + \dots a_k n^k\right) - \boxed{\left(a_0 - a_1 + a_2 \mp \dots + (-1)^k a_k\right)}$$
$$= a_1(n+1) + a_2(n^2-1) + a_3(n^3+1) + \dots + a_{2i}(n^{2i}-1) + a_{2i+1}(n^{2i+1}+1) + \dots$$

is always divisible by $n+1$, and hence by d too.

Hence the answer:

The alternating sum of the digits of the base n representation of a number (as above in the box) is divisible by d if and only if the number itself is divisible by d.

Remark: Technically speaking, we need to start this alternating sum from the last digit and go "from right to left". However, if we are only checking divisibility (and not the remainder in general), then we can start from the first digit a_k: even if the k is odd (the total number of digits $k+1$ is even), then $a_0 - a_1 + a_2 \mp \dots - a_k$ is divisible by d if and only if $-\left(a_0 - a_1 + a_2 \mp \dots - a_k\right) = a_k - a_{k-1} \pm \dots - a_0$ is divisible by d.

When solving this question, it is a good idea to recall test for divisibility by 11. Of course this is a particular case of E) when $n = 10$, i.e. a decimal system.

Now we can state answers in particular cases:

D) A number is divisible by 3 if the alternating sum of its digits in the binary (base 2) system is divisible by 3. For example, 21 is written as 10101_2, and the alternating sum is $1-0+1-0+1$ is divisible by 3.

F) A number is divisible by 5 if the alternating sum of its digits in the base 9 system is divisible by 5. For example, 25 is written as 27_9, and $7-2$ is divisible by 5. Taking a bigger number, say 645, we get that in the base 9 system it is 786_9, and the alternating sum is $7-8+6=5$ is divisible by 5.

3 More advanced questions:

Question 13. *Alice multiplied some real number by* 10 *and got a prime number. Bob multiplied the same real number by* 13 *and ... the result was also a prime number!*

Find out, in which base n *system (the same) they were multiplying. Is such* n *unique? Why?*

Solution:

Let n be the base of the number system. Assume this real number is x. As usual, $10_n = n$ and $13_n = n+3$. Hence we have that nx and $(n+3)x$ are two prime numbers, say p and q. We do not know what $\boldsymbol{x}$ is, but $\dfrac{\boldsymbol{q}}{\boldsymbol{p}} = \dfrac{(\boldsymbol{n}+\mathbf{3})\boldsymbol{x}}{\boldsymbol{n}\boldsymbol{x}} = \dfrac{\boldsymbol{n}+\mathbf{3}}{\boldsymbol{n}}$, or, equivalently, $\boxed{\boldsymbol{q}\boldsymbol{n} = (\boldsymbol{n}+\mathbf{3})\boldsymbol{p}}$. Since $\boldsymbol{p} \neq \boldsymbol{q}$ and $\boldsymbol{qn}$ is divisible by $\boldsymbol{p}$, we conclude that $\boldsymbol{n}$ is divisible by $\boldsymbol{p}$, so $\boxed{\boldsymbol{n} = \boldsymbol{kp}}$ for some positive integer $\boldsymbol{k}$. Hence $\boldsymbol{q}(\boldsymbol{kp}) = (\boldsymbol{kp}+\mathbf{3})\boldsymbol{p}$, or, after dividing by $\boldsymbol{p}$, $\boldsymbol{qk} = \boldsymbol{kp}+\mathbf{3}$. Now we deduce that $\mathbf{3}$ is divisible by $\boldsymbol{k}$, so $\boldsymbol{k} = \mathbf{1}$ or $\boldsymbol{k} = \mathbf{3}$.

If $\boldsymbol{k} = \mathbf{1}$, then $\boldsymbol{qk} = \boldsymbol{kp}+\mathbf{3}$ implies $\boldsymbol{q} = \boldsymbol{p}+\mathbf{3}$, and the only two primes which differ by an <u>odd</u> number $\mathbf{3}$ are $\mathbf{2}$ and $\mathbf{5}$. However in this case $\boldsymbol{n} = \boldsymbol{kp} = \boldsymbol{p} = \mathbf{2}$ which is not possible as we were talking about number $\mathbf{13}$ - the digit $\mathbf{3}$ is not in the base $\mathbf{2}$ number system!

If $\boldsymbol{k} = \mathbf{3}$, then $\boldsymbol{qk} = \boldsymbol{kp}+\mathbf{3}$ implies $\mathbf{3}\boldsymbol{q} = \mathbf{3}\boldsymbol{p}+\mathbf{3}$, so $\boldsymbol{q} = \boldsymbol{p}+\mathbf{1}$, and the only two primes which differ by an <u>odd</u> number $\mathbf{1}$ are $\mathbf{2}$ and $\mathbf{3}$. So $\boldsymbol{p} = \mathbf{2}$, $\boldsymbol{q} = \mathbf{3}$ and $\boldsymbol{n} = \boldsymbol{kp} = \mathbf{6}$.

Let us check: $\boldsymbol{x} = \boldsymbol{p}/\boldsymbol{n}$ is a third. Multiply by $\mathbf{10_6}$, i.e. by six, and get two; multiply by $\mathbf{13_6}$, i.e. by nine, and get three.

Question 14. *What is the smallest number of weights necessary to be able to weigh any number of grams from 1 to 100 on the balance scales with 2 weighing dishes, if weights can only be placed on one of the dishes?*

Solution:

Any number between one and one hundred can be written in the binary (base $\mathbf{2}$ system). Therefore if we have weights $\mathbf{1}$, $\mathbf{2}$, $\mathbf{4}$, $\mathbf{8}$, $\mathbf{16}$, $\mathbf{32}$ and $\mathbf{64}$, then we can make any number between $\mathbf{1}$ and $\mathbf{100}$: looking at its binary decomposition a $\mathbf{1}$ tells us to take a weight and a $\mathbf{0}$ not to take. For example, $\mathbf{50_{10}} = \mathbf{110010_2}$, so writing it from right to left and adding, if necessary, zeros, we get

64	**32**	**16**	**8**	**4**	**2**	**1**
0	1	1	0	0	1	0
-	✓	✓	-	-	✓	-

$\mathbf{32}+\mathbf{16}+\mathbf{2} = \mathbf{50}$.

We cannot do with six weights, since using just six weights one can weigh no more than $\mathbf{2^6} - \mathbf{1} = \mathbf{63} < \mathbf{100}$ different weights (each weight either participates or does not participate in weighing and we need at least one).

Question 15. *What is the smallest number of weights necessary in order to be able to weigh any number of grams from 1 to 100 on the balance scales, if weights can be placed on both dishes?*

Solution:

In solving this question we need the following interesting property of the ternary (base $\mathbf{3}$) number system:

Any natural number can be represented as a difference of two numbers whose representation in the ternary number system contains only $\mathbf{0}$s and $\mathbf{1}$s.

For the proof, we need to write down the original number in the ternary number system and construct the required numbers bitwise from right to left. In this case, if the resulting numbers both have $\mathbf{1}$s in the same column, these two digits they can be replaced by $\mathbf{0}$.

Now it is clear that it is enough to have $\mathbf{5}$ weights with weights $\mathbf{1}$, $\mathbf{3}$, $\mathbf{9}$, $\mathbf{27}$ and $\mathbf{81}$ (each number $\boldsymbol{w}$ between $\mathbf{82}$ and $\mathbf{100}$ can be written as $\mathbf{81} + \boldsymbol{u}$ with $\mathbf{1} \leq \boldsymbol{u} \leq \mathbf{19}$, and $\boldsymbol{u}$ can be represented as a difference between weights $\mathbf{1}$, $\mathbf{3}$, $\mathbf{9}$, $\mathbf{27}$).

Four weights are obviously not enough, because with their help you can weigh not more than $\mathbf{3^4 - 1 = 80 < 100}$ different weights (each weight is either on the left side of the scale or on the right, or does not participate in the weighing).

Question 16. *The Queen of Hearts thinks of three two-digit numbers: $\boldsymbol{A}$, $\boldsymbol{B}$ and $\boldsymbol{C}$. Alice should give the queen three numbers: $\boldsymbol{x}$, $\boldsymbol{y}$ and $\boldsymbol{z}$ after which the Queen will give Alice the sum $\boldsymbol{Ax + By + Cz}$. Alice must guess the numbers $\boldsymbol{A}$, $\boldsymbol{B}$ and $\boldsymbol{C}$ from her first attempt, otherwise her head will be chopped off. How can Alice be saved?*

Solution:

Alice needs to give the following numbers to the Queen: $\boldsymbol{x} = \mathbf{1}$, $\boldsymbol{y} = \mathbf{100}$ and $\boldsymbol{z} = \mathbf{10000}$. Then, as each of $\boldsymbol{A}$, $\boldsymbol{B}$ and $\boldsymbol{C}$ is a two-digit number, in the answer $\boldsymbol{Ax + By + Cz}$ she will see the numbers $\boldsymbol{A}$, $\boldsymbol{B}$ and $\boldsymbol{C}$ written consecutively.

Question 17. *A) Prove that from the set $\{\mathbf{0, 1, 2, \cdots, 3^k - 1}\}$, it is possible to choose $\mathbf{2^k}$ numbers so that none of them is the arithmetic mean of the other two chosen numbers.*

B) Prove that from the set $\{\mathbf{0, 1, 2, \cdots, \frac{1}{2}(3^k - 1)}\}$, it is possible to choose $\mathbf{2^k}$ numbers so that none of them is the arithmetic mean of the other two chosen numbers.

Solution:

We give a solution to B, as these numbers will work for A too.

We use the ternary (base $\mathbf{3}$) number system. The maximal number has the following representation in the ternary system: $\mathbf{\frac{1}{2}(3^k - 1)} = \underbrace{\mathbf{11\ldots11}}_{\boldsymbol{k}}{}_{\mathbf{3}}$.

We will assume that the ternary record of <u>each</u> of these numbers consists of exactly $\boldsymbol{k}$ digits (if necessary, fill the empty high-order bits with zeros). We now choose those numbers whose ternary record contains only the digits $\mathbf{0}$ and $\mathbf{1}$: for instance $\mathbf{1001_3 = 28_{10}}$ is good for us but $\mathbf{121_3 = 16_{10}}$ is not.

There are exactly $\mathbf{2^k}$ such numbers. We show that this is the desired collection. Suppose the contrary: among the selected numbers there are three different numbers $\boldsymbol{x, y, z}$, satisfying the equation $\boldsymbol{x + y} = \mathbf{2}\boldsymbol{z}$. Since the numbers $\boldsymbol{x}$ and $\boldsymbol{y}$ differ at least in one digit, and all digits are $\mathbf{0}$s and $\mathbf{1}$s, in the ternary representation of their sum $\boldsymbol{x + y}$ in this digit we must have $\mathbf{1}$. However, in the record of the number $\mathbf{2}\boldsymbol{z}$ there are only $\mathbf{0}$ and $\mathbf{2}$.