STRANGE DISTANCE FUNCTIONS AND STRANGER CIRCLES

For the next couple weeks, we will explore some interesting ways of measuring distances between points in the plane. We will especially be looking for curious or surprising behavior.

Supplies: please bring some colored pencils next time. Some extra graph paper might be helpful as well.

1 Review (?) of some properties of standard Euclidean distance

To review, and also to give ourselves something "normal" to compare to, let's recap some of what happens with standard Euclidean distance.

1. Plot the points $B = (-1, 4)$, $M = (4, -3)$, and $C = (5, 5)$ on graph paper.
2. Can you find the distance between each pair of points? What famous theorem are we using?
3. What is the geometric definition of a circle?
4. Sketch a circle of radius 1 about the origin $\mathcal{O}$. (For today, you can sketch by hand, or you can use your compass if you are already familiar with it and can make circles pretty quickly.) Repeat with radius 2, radius 3, etc. so that you have a collection of concentric circles.
5. Also sketch circles with radius 1, 2, 3, 4, 5 etc about point M.
6. Did you find any points that are equidistant from M and $\mathcal{O}$?

2 The taxicab metric

For this section, imagine a city (huge, practically infinite) whose streets are on a nice square grid, and they are all two-way streets. We'll sometimes call this the integer grid You are a taxicab driver and can only drive on the streets. Let's think about how to measure driving distances in this setup. (Since you are too young for driving cars, let's assume you have a bicycle taxi that pulls a small cart holding passengers.)

7. Plot the points $B = (-1, 4)$, $M = (4, -3)$, and $C = (5, 5)$ below. Suppose each square on the grid is 1 block by 1 block (as in city blocks).
8. Draw several paths you could take from B to C. Is there a shortest possible path from B to C, without breaking the law? Multiple shortest paths with the same smallest number of blocks biked?
9. Let's say that the shortest taxicab path between two points is our taxicab distance. Find the taxicab distance from B to M, denoted BM_T. Is this the same as the taxicab distance from M to B, denoted MB_T?
10. Same question as above, but for C and M.
11. Which is smaller, $MC_T + CB_T$ or MB_T?
12. Can you think of three points X, Y, and Z so that $XY_T + YZ_T = XZ_T$?

13. How about three points X, Y, and Z so that $XY_T + YZ_T < XZ_T$? Show the details of your example if you can find one, or if you think no such example is possible, try to explain why.
14. Next, let's look at "taxicab circles" about the origin. First mark all points on the grid whose taxicab distance from the origin is 1 block. Next, using a different color if you can, mark all points whose taxicab distance from the origin is 2. Next, distance 3 blocks, distance 4 blocks, etc. Do your taxicab circles look like what you normally think of as circles?
15. Does it make sense to consider a radius which is not an integer? Try radius 2.5, for example – can you find all points on the integer grid which are 2.5 blocks from the origin?.
16. Try to find all points which are taxicab equidistant from the points $(0, 0)$ and $(4, 0)$.
17. Try to find all points which are taxicab equidistant from the points $(0, 0)$ and $(4, 2)$.
18. Try to find all points which are taxicab equidistant from the points $(0, 0)$ and $(4, 4)$.
19. Try to find all points which are taxicab equidistant from the points $(0, 0)$ and $(3, 5)$.

3 Comparing Euclidean and Taxicab distances

Here are a few more abstract questions to think about:

20. Can you find pairs of points whose taxicab distance is equal to their Euclidean distance?
21. If two points on the integer grid have the same taxicab distance as each other, must they also have the same Euclidean distance as each other? (The Euclidean distance does not need to match the taxicab distance.)
22. If two points on the integer grid have the same Euclidean distance as each other, must they also have the same taxicab distance as each other? (The Euclidean distance does not need to match the taxicab distance.)
23. With the standard Euclidean distance, the set of points equidistant from two points A and B is their perpendicular bisector. Do you think this is also the case with the taxicab distance? Explain your reasoning.
24. Do you think it would make sense to have a distance function on the full plane, where only vertical and horizontal motion is allowed (no diagonals). We would have to change our story, but perhaps the pure math is still interesting. What do circles look like?

4 Properties of a metric, AKA distance function

DEFINITION: A function d which takes two points as inputs is a **metric** or **distance function** on the plane if (and only if) the following three properties hold. *(These are informal statements of the properties, and we will fill in the more rigorous ones.)*

(A) The distance from any point to itself is 0, and the distance between any two different points is positive.

(B) Distance is symmetric.

(C) Distance satisfies the triangle inequality.

We have not tried to formally prove these properties about the taxicab distance, but we have seen some evidence that they may be true. For those of you who have some experience writing proofs, try to show that all of these hold for the taxicab distance we have been discussing.

5 Some other potential distance functions

Next time, we will discuss more of these. I don't want to give you the details now; instead brainstorm what you think they might be, based on the names.

1. Post Office metric
2. Chess King metric
3. Teleportation metric
4. Infinitely Long and Tall Hotel with Only One Elevator metric