PARITY AND INVARIANTS

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1. Numbers 1,2,3,…, 3001 are written on a board. You are allowed to replace any two of these numbers by one number, which is either the sum or the difference of these two numbers. You perform this operation 3000 times. How many numbers are left on the board? Can that number be 6000?

2. There are 5 red marbles and 6 green marbles in a jar. Pascal plays a strange game. He removes two marbles at a time, with the following rules:

a. If the marbles are both green, he puts one green marble back.

b. If there is one marble of each color, he puts one red marble back.

c. If the marbles are both red, he puts one green marble (from his infinite supply of different marbles) back in the jar.

At the end, there will be one marble left. Which color is it?

3. A grasshopper jumps along a line. His first jump takes him 1 cm, his second 2cm, and so on. Each jump can take him to the right or to the left. Is it possible that after 3002 jumps the grasshopper returns to the point at which he started?

4. A witch caught 100 dwarfs and she decided to give them a test. If they pass, they will all go free. Otherwise, they will all become witch's servants for life. The test goes as follows: the dwarfs stand in a line, all facing forward. The witch puts either red or blue hat on each dwarf. Each dwarf can only see the colors of the hats in front of him; he can't see his the color of his own hat or the hats of the dwarfs behind him. Then, in any order they want, each dwarf guesses the color of the hat on his head, and says "Red" or "Blue." Other than that, the dwarfs cannot speak; they are not allowed to send any signals like coughing, throwing pieces of paper, touching each other, or anything of that sort. To pass the test, no more than one of them may guess incorrectly. If the dwarfs can agree on their strategy beforehand, can they be sure that they will all go free?

5. Of 101 coins, 50 are counterfeit, and differ from the genuine coins in weight by 1 gram. Jaime has a balance scale which shows the difference in weights between the objects placed in each pan. He chooses one coin, and wants to find out in one weighing whether it is counterfeit or not. Can he do this?

6. You start with an 8-by-8 chessboard. For some reason, your chessboard is missing two opposite corners, as shown below. You have 31 dominoes, and you want to tile what's left of the chessboard with these dominoes. (A domino is a 2-by-1 tile and so it covers exactly two squares of the chessboard.) Can you do it?

7. Is it possible to tile a 10-by-10 grid with 25 1-by-4 tiles?

8. A rectangular floor is covered by 2-by- 2 and 1-by-4 tiles. One tile got smashed, but we have one more tile of the other kind available. Can we retile the floor perfectly?

9. A dragon has 100 heads. A knight can cut off 15, 17, 20, or 5 heads, respectively, with one blow of his sword. In each of these cases, 24, 2, 14, or 17 heads grow on its shoulders. If all heads are blown off, the dragon dies. Can the dragon ever die?

10. Prince Ivan is on his way to free his younger brother imprisoned by their enemies when he is confronted by an evil dragon. Ivan has two magic swords. With each blow of the first sword Ivan can cut off 21 heads of the dragon. With each blow of the second sword he can cut off 13 heads but after that the dragon will grow 594 new heads. In order to continue his journey, Ivan must cut off all the dragon's heads. Could Ivan succeed if originally the dragon had 100 heads?

11. (a) A mad veterinarian has invented an animal transmogrifying machine. If you put in two cats or two dogs, then one dog comes out of the machine. If you put in one cat and one dog, then one cat comes out. The veterinarian's goal is to end up with only one cat and no other animals. If he starts with 3 cats and a dog, can he achieve his goal? What if he starts with 13 cats and 10 dogs? Could you describe all starting collections which allow the veterinarian to achieve his goal?

(b) The veterinarian's old machine breaks. Now he has cats, dogs, and mice. The new transmogrifying machine can take in any two different animals and then out comes the third animal. Can the veterinarian achieve his goal of ending up with exactly one cat and no other animals if he starts again with 3 cats and 1 dog? With 4 of each animals?

(c) What happens if you can now use the machine going in any direction, forward or backward at your wish – ether putting in any two different animals and getting out one animal of the third type, or putting in one animal of any kind and getting out two new animals, one of each of the two other type – e.g., if you put in a dog and a mouse, you receive a cat?

12. On a planet far away, the only inhabitants are chameleons. They come in three colors: green, yellow, and red. If two chameleons of the same color meet nothing changes. But if two chameleons of different colors meet, they both change to the third color. If there were 4 green, 5 yellow, an 5 red chameleons to begin with, is it possible that all chameleons will eventually be of the same color? What if there were 13 green, 15 yellow, and 17 red chameleons to begin with? Try some other initial numbers; what can you observe?

13. (a) Numbers 1,2,3,…,100 are written on the board. Every minute, we erase any two of them (say, x and y), and write their sum, $x + y$. What happens in the long run?

(b) What if in the previous problem $x+y$ is replaced by xy?

(c) What if it is replaced with $xy+x+y$?

14. An infection spreads among the squares of a 10-by-10 grid in the following way: if a square has two or more infected neighbors, then it becomes infected itself (a neighbor being a square which shares an edge). If initially there were 9 infected squares, is it possible that all 100 squares will eventually become infected?