Generating Functions

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September 6, 2016

1 Ordinary Generating Functions

Definition. The *ordinary generating function* for a sequence $(a_0, a_1, a_2, ...)$ is the sum

$$\sum_{n=0}^{\infty} a_n x^n.$$

Problems

1. Alice has two standard dice, and Bob has two weird dice, one with the numbers 1, 2, 2, 3, 3, and 4, and the other with 1, 3, 4, 5, 6, and 8. Each person roles both dice and finds the sum of the two numbers. Who is more likely to get each sum?

2. Prove that the generating function for choosing n items from the union of the sets A and B is the product of the generating functions for choosing n items from A or from B.

3. Find a formula for coefficients in the product of the generating functions for the sequences

   $$(a_0, a_1, a_2, ...),\ (b_0, b_1, b_2, ...)$$

   (These are called *convolutions*.)

4. Find the generating function for the nth row of Pascal's triangle. What identity does the previous problem give you?

5. Find the number of ways to fill a bag of fruits given the following constraints:

   (a) The number of apples is even.
   (b) The number of bananas is a multiple of 5.
   (c) There are at most four oranges.
   (d) There is at most one pear. (*MIT*)

6. Prove that

$$\frac{1}{(1-x)^n} = \sum_{k=0}^{\infty} \binom{n-1+k}{k} x^k.$$

7. Find a formula for the sequence corresponding to the derivative of a given generating function.

8. Find a closed form for the generating functions for the squares and for the cubes

   (a) using the generating functions for distributions.

   (b) using derivatives.

9. Prove that the generating function for the partitions is

$$\prod_{n=1}^{\infty} \frac{1}{1-x^n}.$$

10. Prove that the number of partitions of n with no repeated parts equals the number of partitions with only odd parts (*Aops*).

11. Prove that the number of partitions of n in which no part appears exactly once equals the number of partitions in which no part is one greater or one less than a multiple of 6 (*Aops*).

12. For a partition Π of n, let $f(\Pi)$ be the number of 1's in Π and $g(\Pi)$ be the number of distinct numbers in Π. Prove that the sum $s(n)$ of $f(\Pi)$ over all partitions Π of n equals the sum $t(n)$ of $g(\Pi)$ over all partitions of n (*USAMO*).

13. (a) Prove that the generating function for the Fibonacci numbers is

$$\frac{x}{1-x-x^2}.$$

    (b) Use this to find a formula for the Fibonacci numbers.

14. (a) Prove that the generating function for the Catalan numbers is

$$\frac{1-\sqrt{1-4x}}{2x}.$$

    (b) Use this to find a formula for the Catalan numbers.

2 Exponential Generating Functions

Definition. The *exponential generating function* for a sequence $(a_0, a_1, a_2, ...)$ is defined as

$$\sum_{n=0}^{\infty} \frac{a_n x^n}{n!}.$$

Problems

1. What is the exponential generating function for subsets of an n element set?

2. Prove that the product of the exponential generating functions for the sequences

$$(a_0, a_1, a_2, ...),\ (b_0, b_1, b_2, ...)$$

   has nth coefficient

$$\sum_{k=0}^{n} \binom{n}{k} a_k b_{n-k}.$$

3. Show that the product of the exponential generating functions for choosing n ordered objects (according to certain rules) from set A or set B is the exponential generating function for choosing n ordered items (under the same rules) from the union of the two sets.

4. What is the generating function for the permutations? How can the identity

$$F(x) = 1 + xF(x)$$

   be interpreted combinatorially?

5. How many sequences n letters can be formed from A, B, and C such that the number of A's is odd and the number of B's is odd? (*Princeton*)

6. How many ways can n people be arranged into pairs? (*Princeton*)

7. A *derangement* is a permutation of n objects with no fixed points.

   (a) What is the exponential generating function for the derangements?

   (b) How many derangements of n objects are there?

8. (a) What is the exponential generating function for functions $A \mapsto \{1, 2, ..., k\}$, where A is an n-element set?

   (b) What about surjective functions?

9. Show that the exponential generating function for the partitions is

$$\sum_{k=0}^{\infty} \frac{(e^x - 1)^k}{k!} = e^{e^x - 1}.$$

10. Prove the *composition principle:* the composition $G(H(x))$ of two exponential generating functions for g-structures and h-structures (where h-structures are nonempty) corresponds to partitions of n into blocks, with an h-structure imposed on each block and a g-structure on the set of blocks.

11. Find the exponential generating function for

    (a) partitions into sets with more than one element.

    (b) partitions with only odd-sized blocks.

12. Find the exponential generating function for preference orderings on n items, where any two items are comparable, but some items may be liked equally well (UCB).

13. Find the exponential generating function for permutations by thinking of the permutations as broken into cycles (UCB).

14. Prove the *derivative principle*: for an exponential generating function $H(x)$ counting h-structures on a set A, the derivative $H'(x)$ counts h-structures on the set A with one extra element.

15. Find the exponential generating function for permutations (interpreted as linear orderings) using a differential equation.

16. Find the exponential generating function for *alternating permutations* of $\{1, ..., n\}$, permutations such that

$$a_1 < a_2 > a_3 < a_4 > ... < a_{n-1} > a_n$$

    (these only exist when n is odd) using a differential equation (UCB).