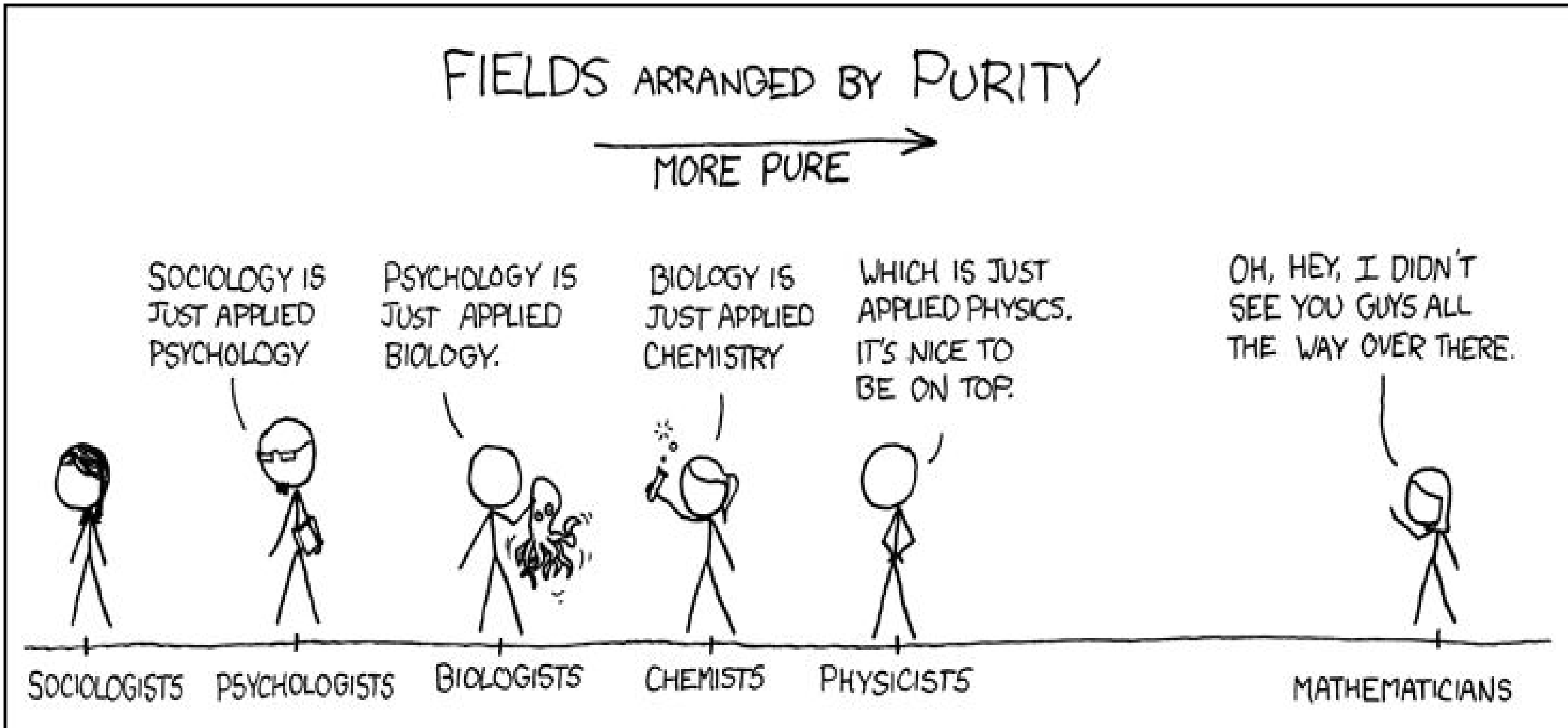
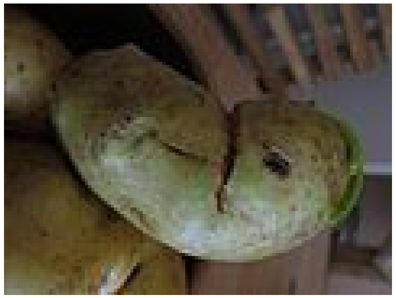
Monstrous groups

Berkeley Math Circle 2016



http://xkcd.com/435/

Symmetry and asymmetry

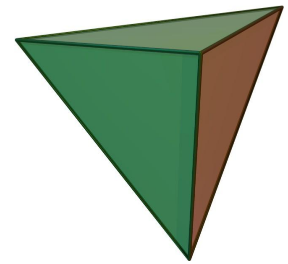








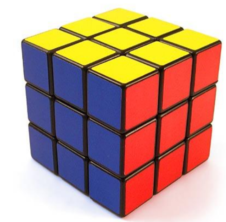
What is a group?



A group is the symmetries of something.

*A tetrahedron has 12=4x3 symmetries:

4=number of faces, 3=symmetries of each face.



* A cube has 24=6x4 symmetries.

6=number of faces, 4=symmetries of each face.

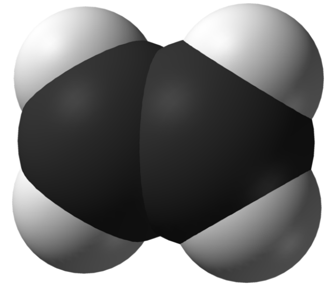
*An octahedron has 24=8x3 symmetries:

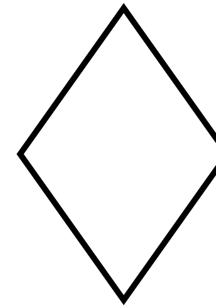
8=number of faces, 3=symmetries of each face.

The number of elements of a group is called its **order.**

The **order of an element** of a group is the number of times you have to repeat it to get back to where you started.

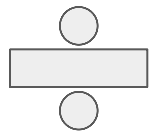
The two groups of order 4





1 symmetry of order 1, 3 of order 2:

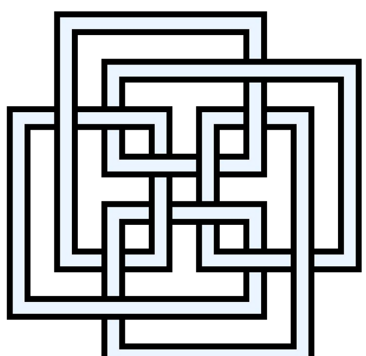
H X Θ Ж Φ 8 □



Klein 4-group

---

1 symmetry of order 1, 1 of order 2, 2 of order 4. Cyclic group.



Tennis ball question

A tennis ball has 4 symmetries (allowing just rotations) corresponding to 4 loops.

Which of the two groups of order 4 is its symmetry group? The one with 2 elements of order 4, or the one with 3 elements of order 2?

Apple

1 trivial symmetry Order 1

2 rotations by 1/5 revolution Order 5

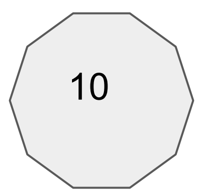
2 rotations by 2/5 revolution Order 5

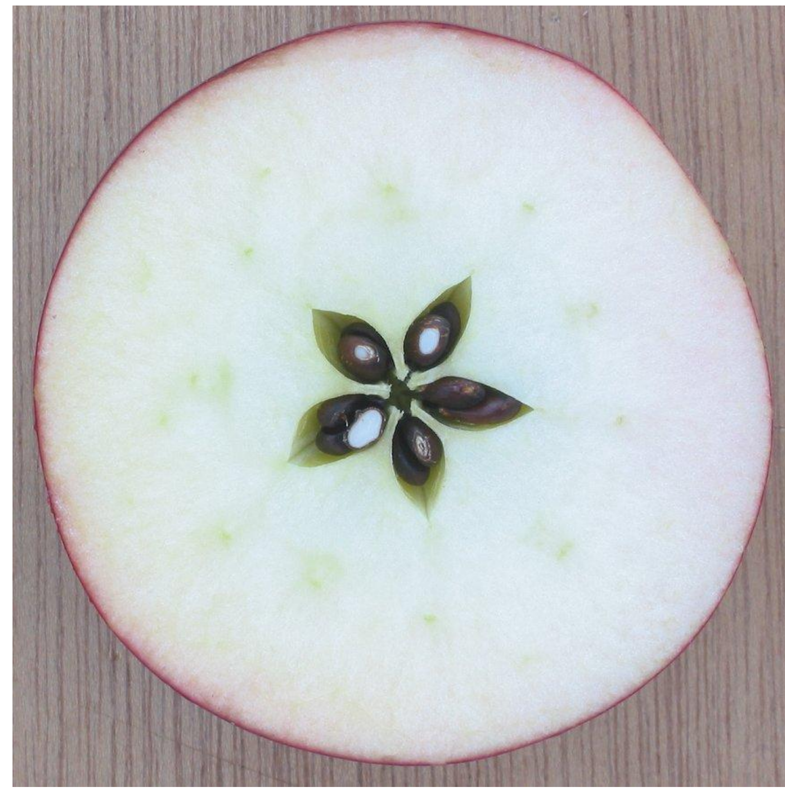
5 reflections Order 2

10 total

Differs from rotations of 10-gon

which has 4 symmetries of order 10



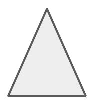


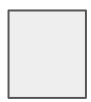
How many groups of small order

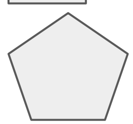
Rotations | Rotations and reflections

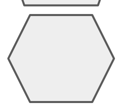
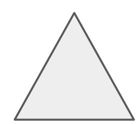
1 group of order 1 F

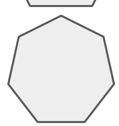
1 group of order 2 S 👧 A B C D E

1 group of order 3 

2 groups of order 4:  

1 group of order 5 

2 groups of order 6:  

1 group of order 7 

5 groups of order 8

2 groups of order 9

Groups of larger order

There are 267 groups of order $64=2^6$

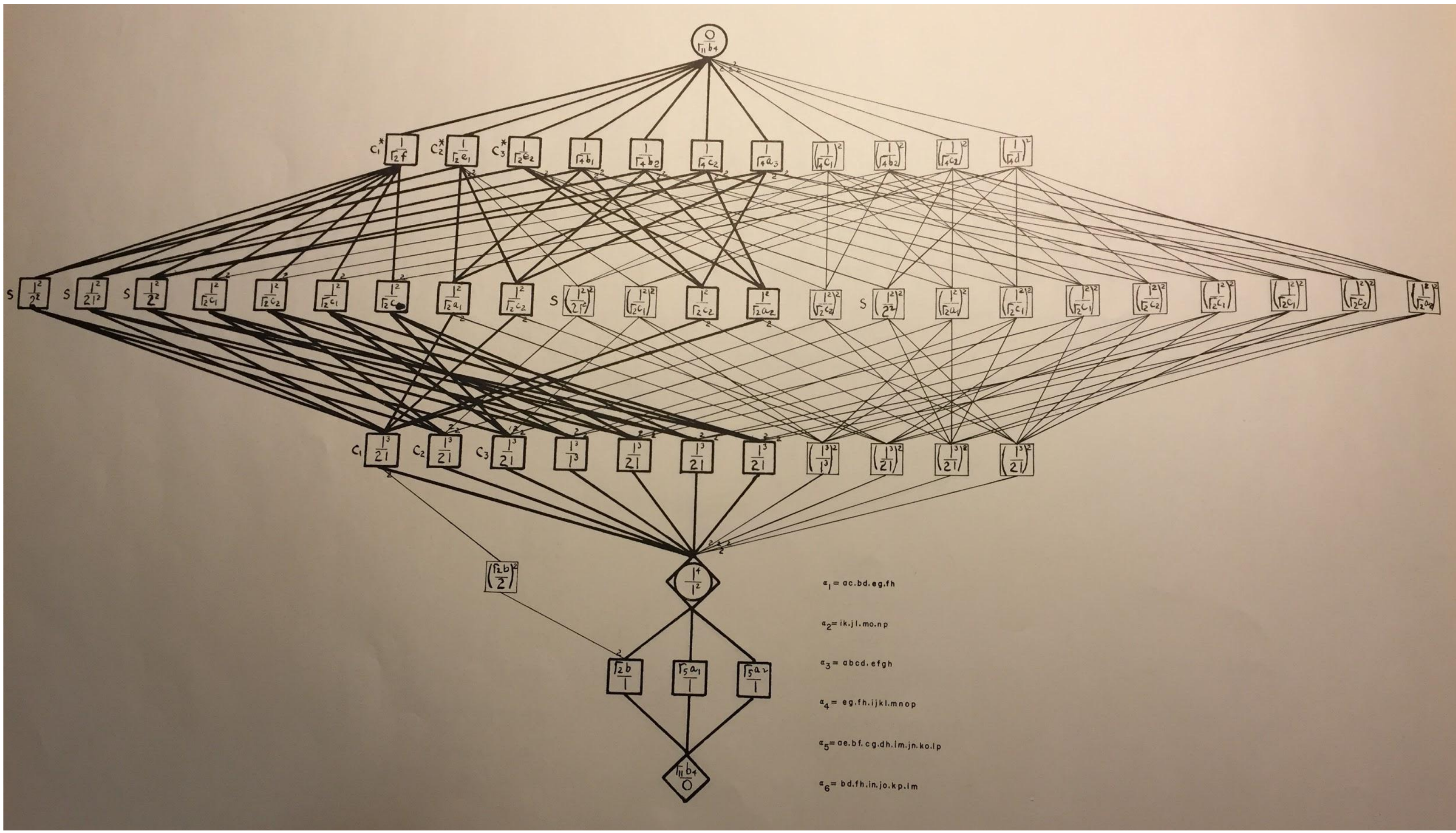
There are 49487365422 groups of order $1024 = 2^{10}$

There is 1 group of order 1000003 (1000003 is prime)

The number of groups of order N depends more on the number of primes in the prime factorization of N than on the size of N.

Conclusion: It is hopeless to classify all groups of order N if N is a product of many primes.

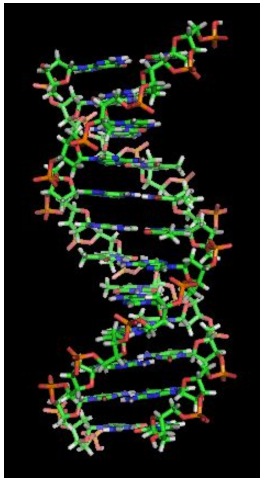
Some subgroups of one of the groups of order 64

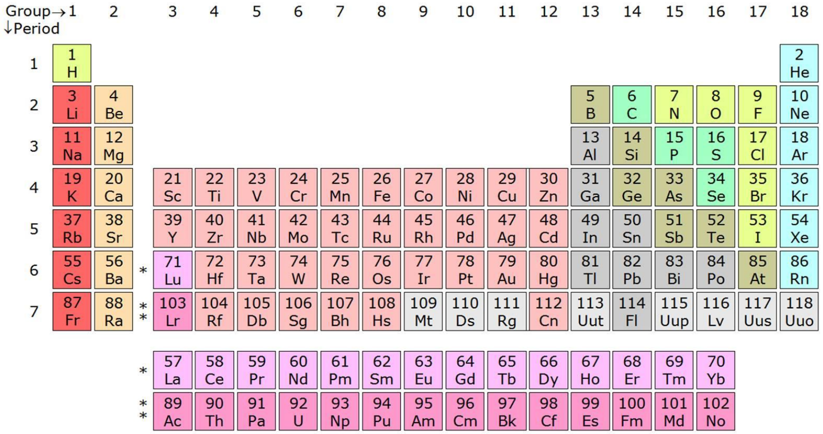


Évariste Galois 1811 - 1832

Atoms and simple groups

It is hopeless to classify all chemical compounds, but we can split compounds into ATOMS and classify the 100 or so atoms.

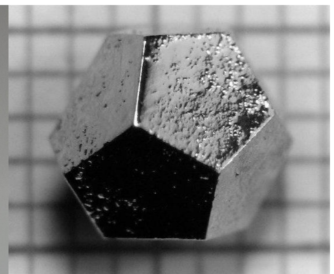
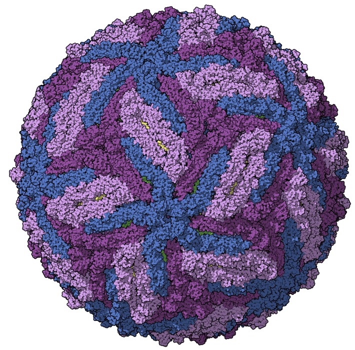
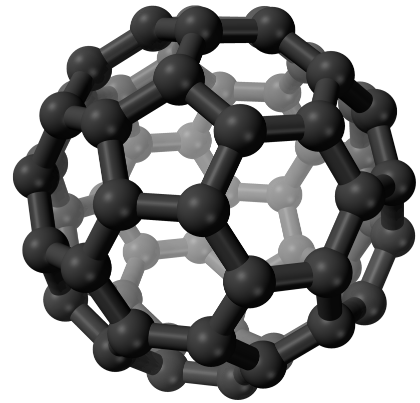


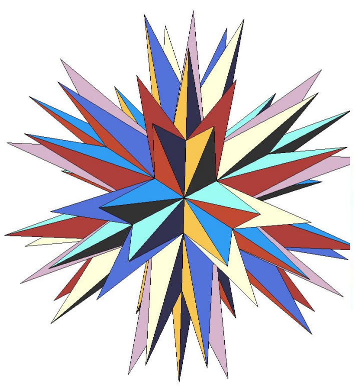
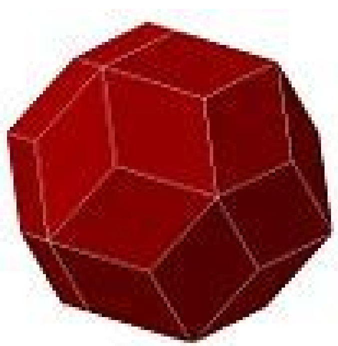
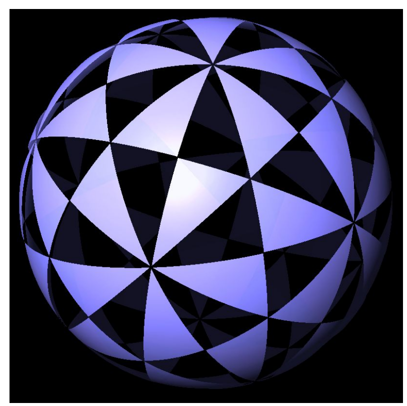
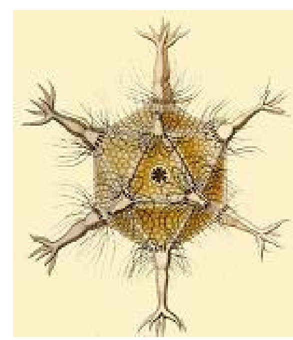


Group→ ↓Period	1	2		3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18
1	1 H																		2 He
2	3 Li	4 Be												5 B	6 C	7 N	8 O	9 F	10 Ne
3	11 Na	12 Mg												13 Al	14 Si	15 P	16 S	17 Cl	18 Ar
4	19 K	20 Ca		21 Sc	22 Ti	23 V	24 Cr	25 Mn	26 Fe	27 Co	28 Ni	29 Cu	30 Zn	31 Ga	32 Ge	33 As	34 Se	35 Br	36 Kr
5	37 Rb	38 Sr		39 Y	40 Zr	41 Nb	42 Mo	43 Tc	44 Ru	45 Rh	46 Pd	47 Ag	48 Cd	49 In	50 Sn	51 Sb	52 Te	53 I	54 Xe
6	55 Cs	56 Ba	*	71 Lu	72 Hf	73 Ta	74 W	75 Re	76 Os	77 Ir	78 Pt	79 Au	80 Hg	81 Tl	82 Pb	83 Bi	84 Po	85 At	86 Rn
7	87 Fr	88 Ra	**	103 Lr	104 Rf	105 Db	106 Sg	107 Bh	108 Hs	109 Mt	110 Ds	111 Rg	112 Cn	113 Uut	114 Fl	115 Uup	116 Lv	117 Uus	118 Uuo
			*	57 La	58 Ce	59 Pr	60 Nd	61 Pm	62 Sm	63 Eu	64 Gd	65 Tb	66 Dy	67 Ho	68 Er	69 Tm	70 Yb		
			**	89 Ac	90 Th	91 Pa	92 U	93 Np	94 Pu	95 Am	96 Cm	97 Bk	98 Cf	99 Es	100 Fm	101 Md	102 No		

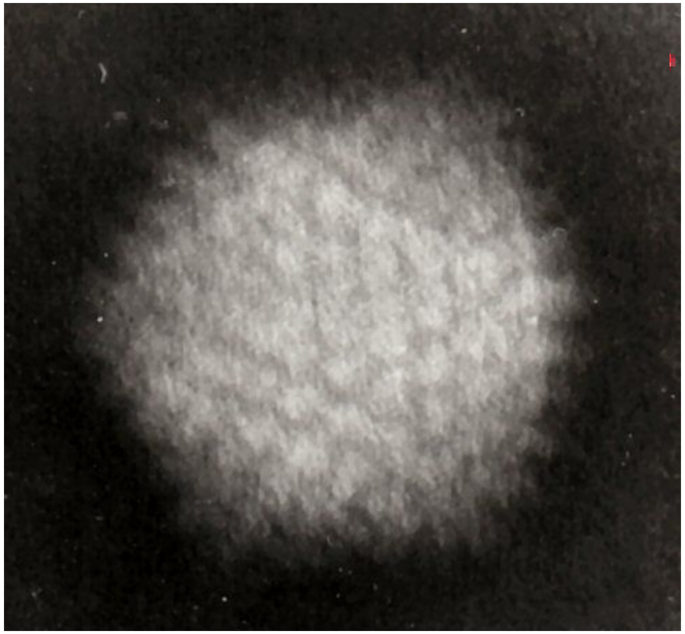
It is hopeless to classify groups, but we can split groups into SIMPLE GROUPS and classify the simple groups.

Icosahedral group: 60 symmetries

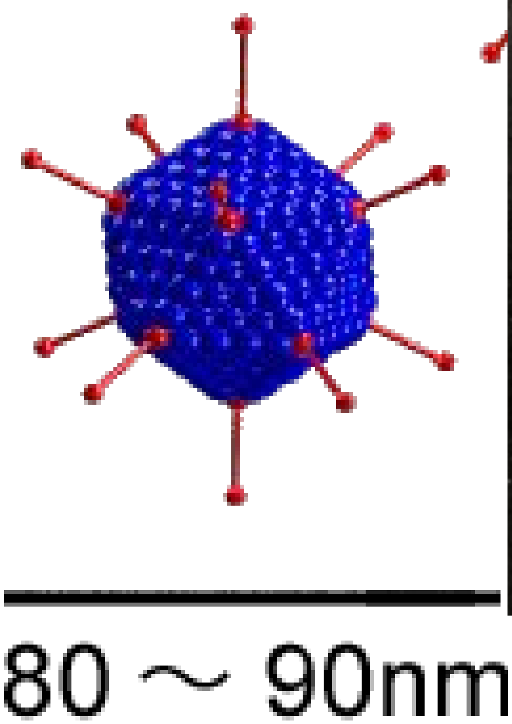




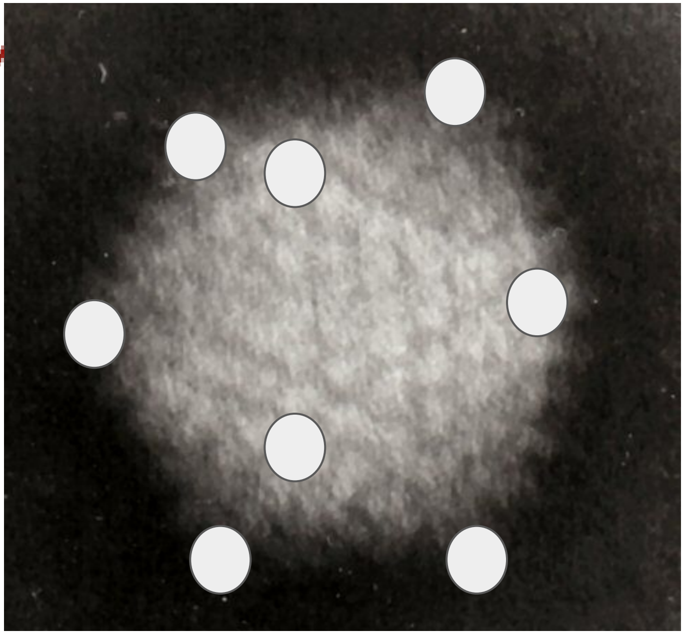






Adenovirus



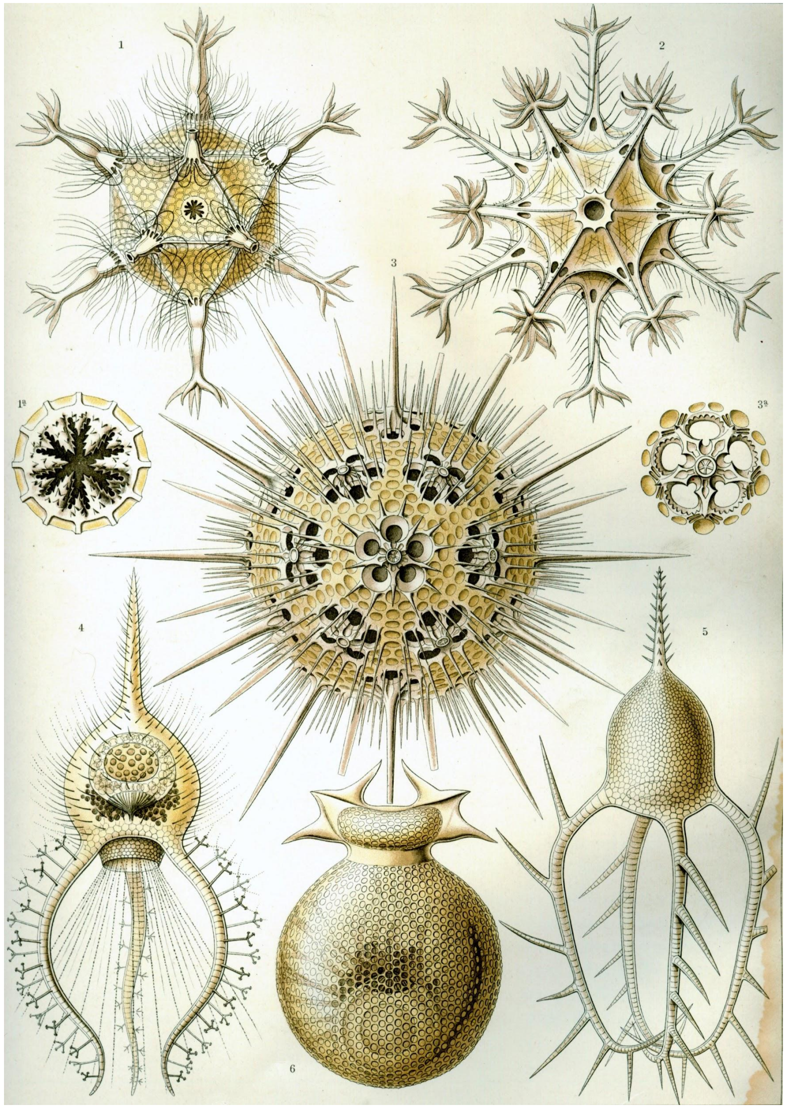
Transmission electron micrograph of adenovirus

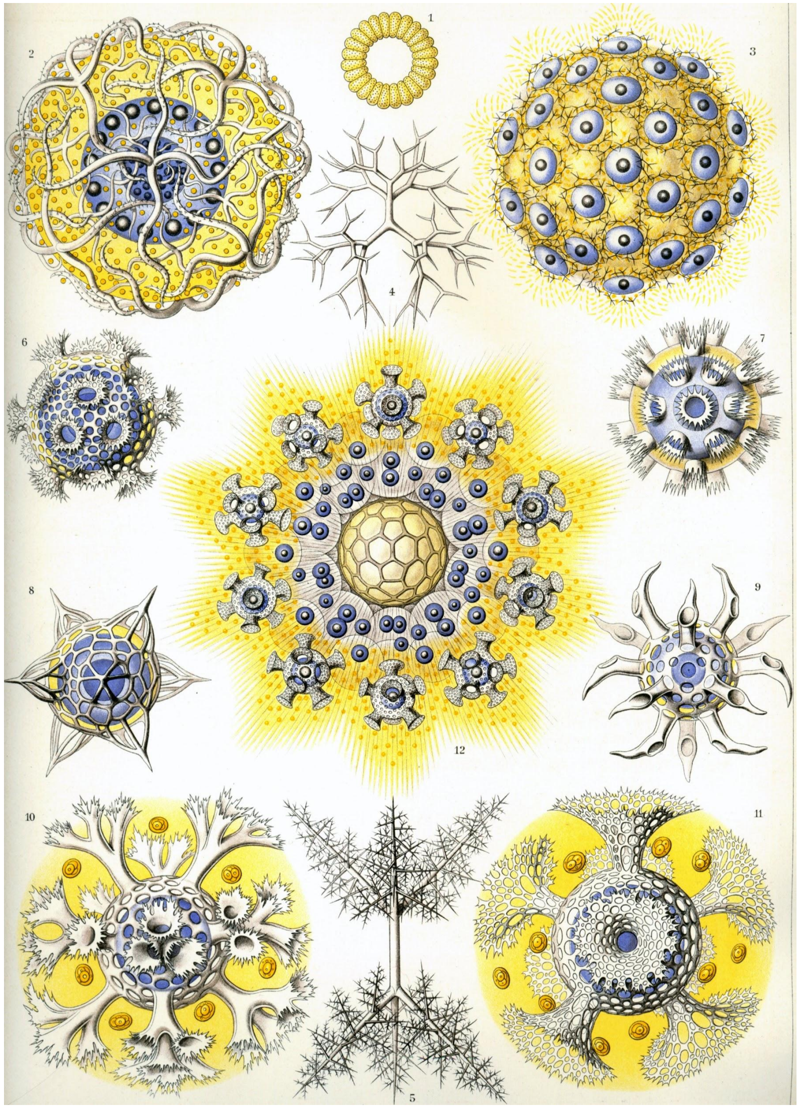




Vertices of the icosahedron marked as white dots

More radiolarians



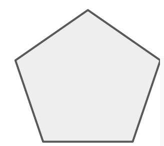


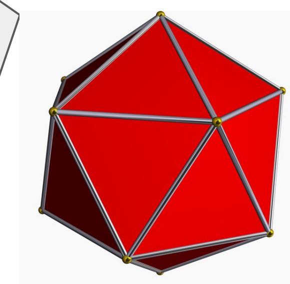
Classification of finite simple groups

Classification of simple groups

Conclusion of classification: There are 18 infinite families of simple groups, and 26 others called the "sporadic groups".

Examples:





*Cyclic groups of prime order 2, 3, 5, 7, 11, 13, ...

*The icosahedral group of order 60. (Smallest non-cyclic)

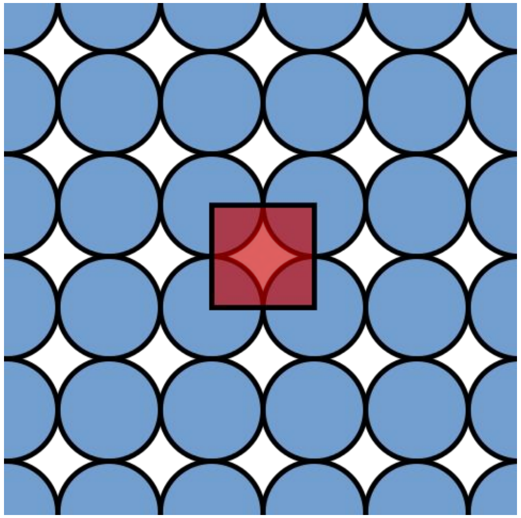
*The Mathieu group M_{11} of order 7920. (Smallest sporadic)

*The monster group of order 808017424794512875886459904961710757005754368000000000

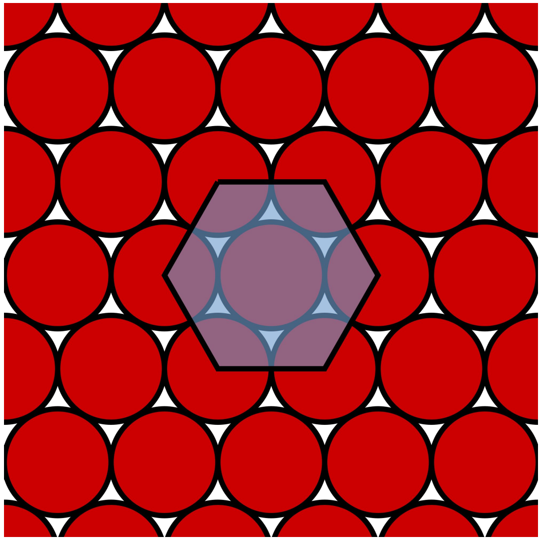
Every finite group can be built out of the simple groups.

Sphere packing in 2 dimensions

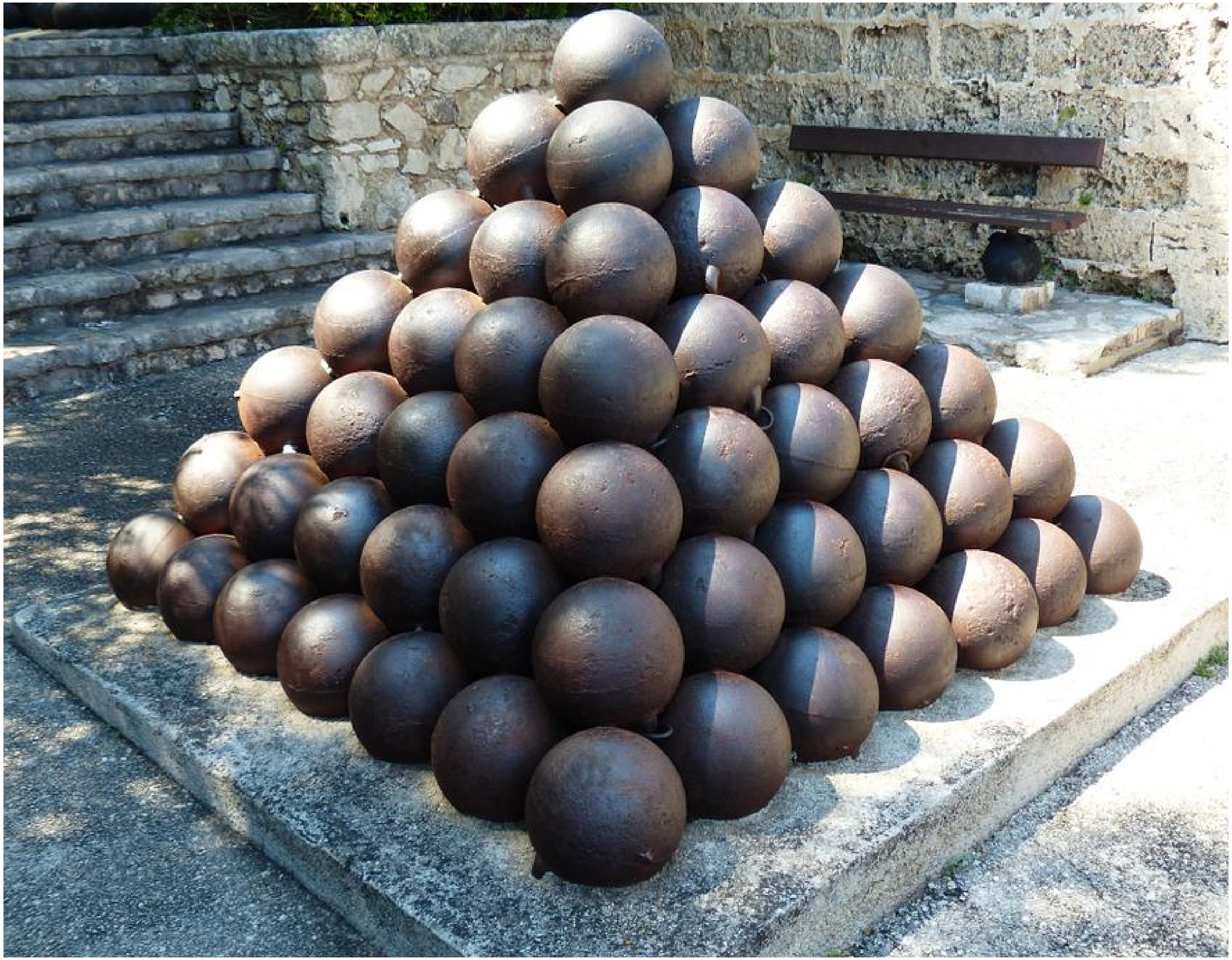
Square packing: good.



Hexagonal packing: better.



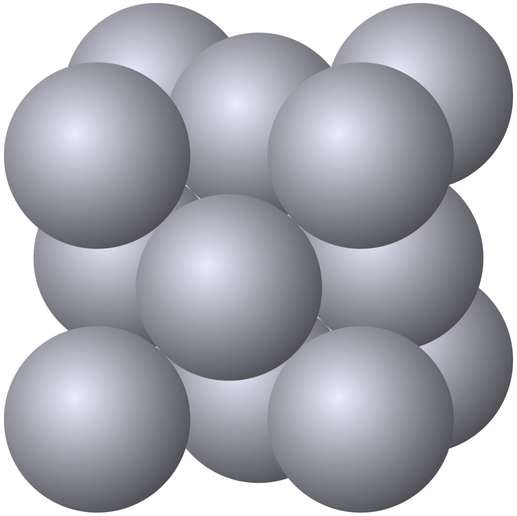
Stack of cannonballs



Sphere packing in 3 dimensions

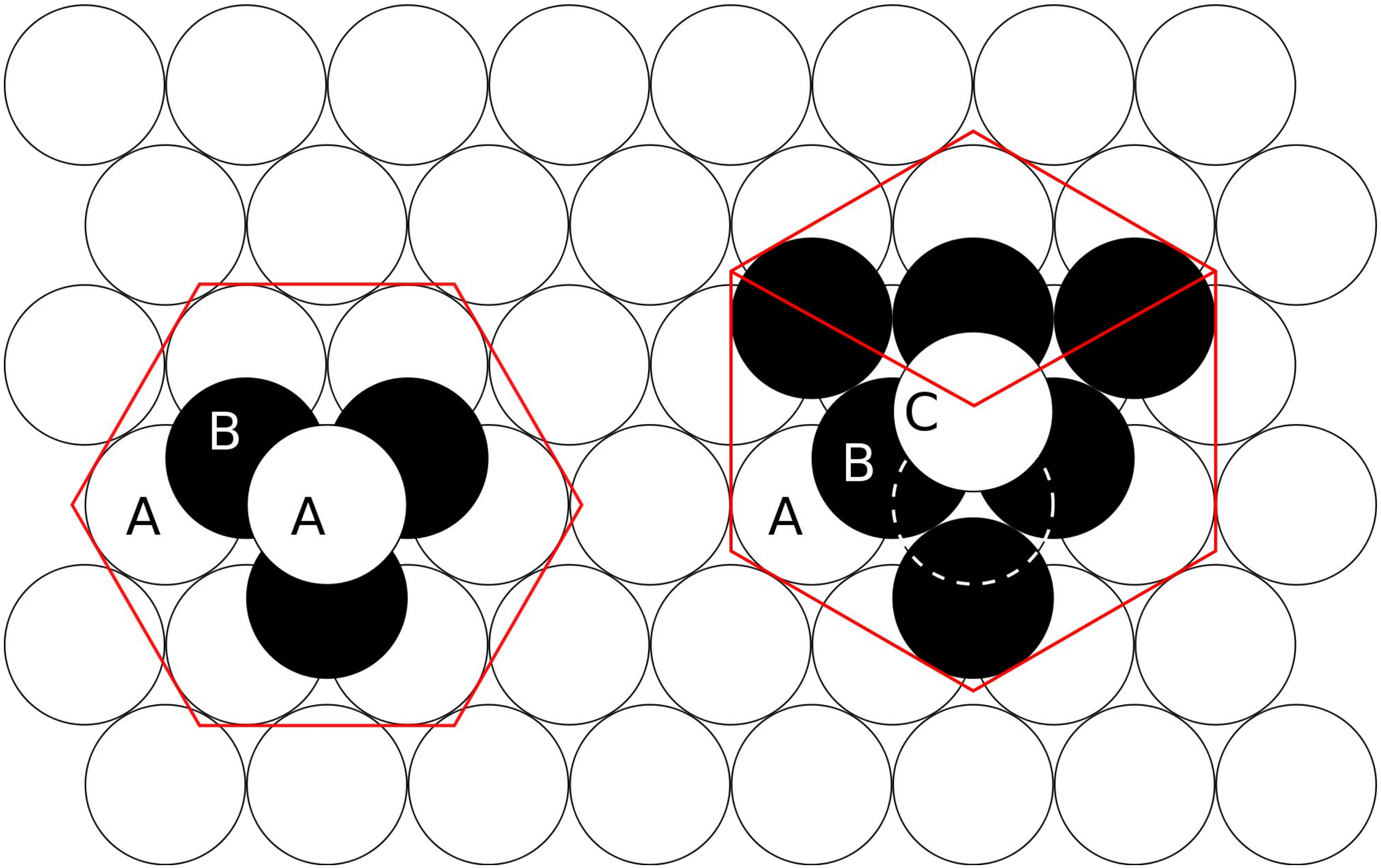
Cubic packing. Obvious, but not best:

Face centered cubic:



Hexagonal close packing:

Face centered cubic (FCC) and Hexagonal close packing (HCP)



HCP: ABABABABAB....   FCC: ABCABCABC...

Best sphere packing in various dimensions

Dimension 1: trivial (unique)

Dimension 2: Hexagonal (unique)

Dimension 3: Face-centered cubic (not unique) (a,b,c), a+b+c even

Dimension 4 to 7: ??? (a,b,c,d), a+b+c+d even?

Dimension 8: E8 lattice (unique) (a,b,c,d,e,f,g,h), sum even, all integers

Dimensions 9 to 23: ??? or all integers+½

DImension 24: Leech lattice (unique)

Dimensions greater than 24: ???

Proofs of best sphere packings in 3, 8 dimensions

Face centered cubic is densest 3-dimensional

sphere packing (Kepler conjecture):

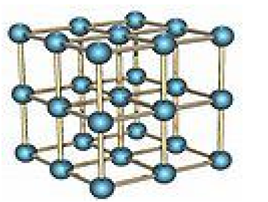
Thomas Hales (1998).

E8 is densest 8-dimensional sphere packing:

Maryna Viazovska (2016).

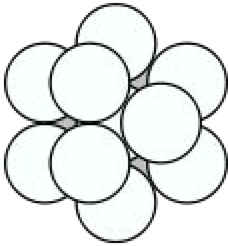
How to describe a sphere packing

Describe sphere packings by giving coordinates of centers of spheres.



Cubic packing: (a,b,c), a,b,c, integers.

6 spheres (±1, 0, 0), (0, ±1, 0), (0, 0, ±1) touching sphere at (0,0,0)



Face centered cubic: (a,b,c), a,b,c, integers, sum a+b+c even

12 spheres (±1, ±1, 0), (0, ±1, ±1), (±1, 0, ±1) touching sphere at (0,0,0)

E8 packing: (a,b,c,d,e,f,g,h), sum even, all integers or all integers+½

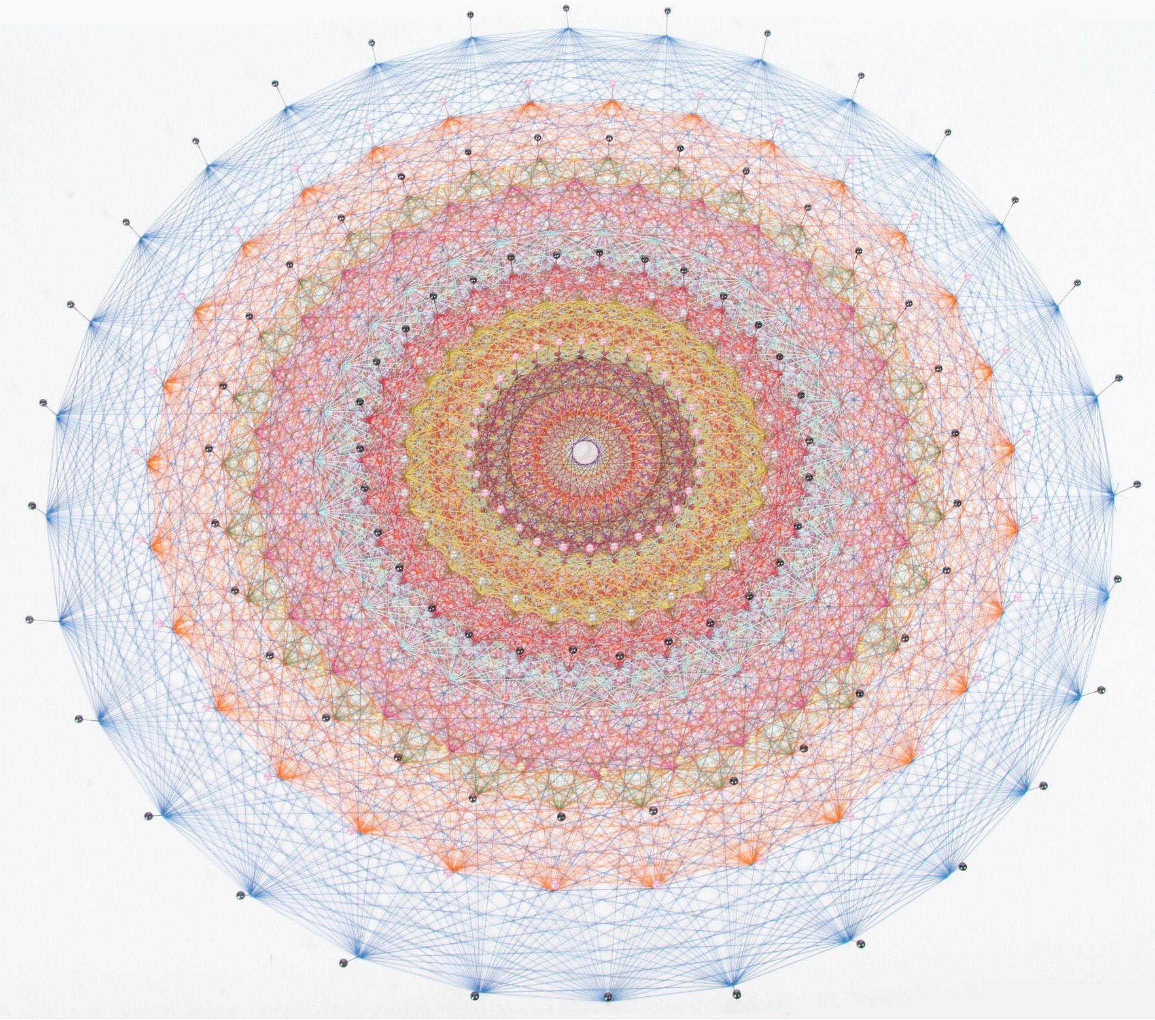
240 spheres touching sphere at (0,0,0)

(±½, ±½, ±½, ±½, ±½, ±½, ±½, ±½) Even sum (128 vectors)

(±1, ±1, 0, 0, 0, 0, 0, 0) and all vectors of similar "shape" (112 vectors)

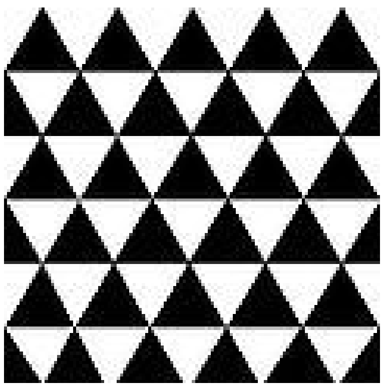
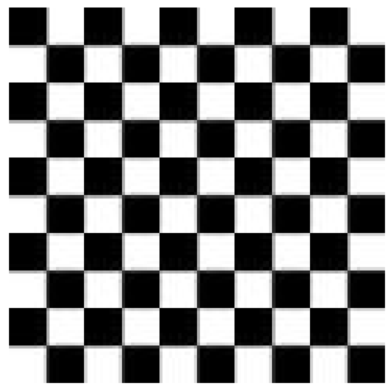
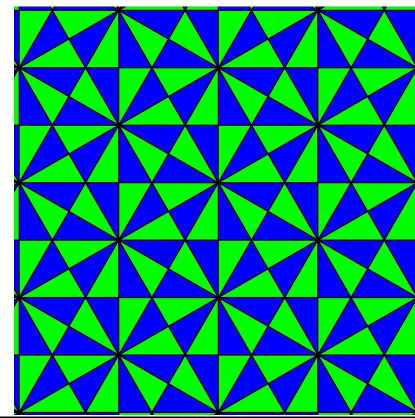
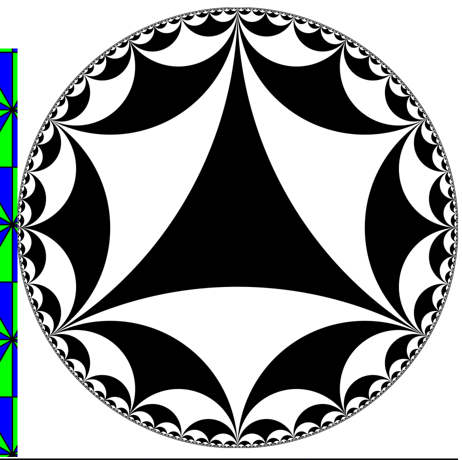
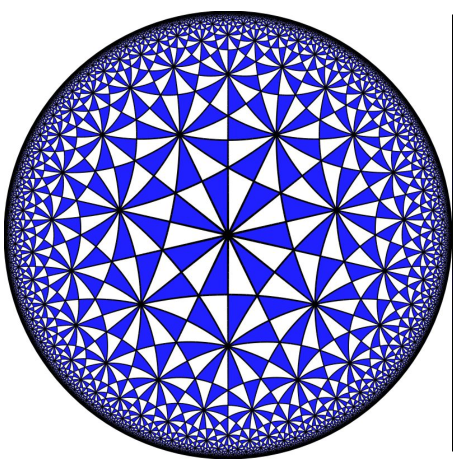
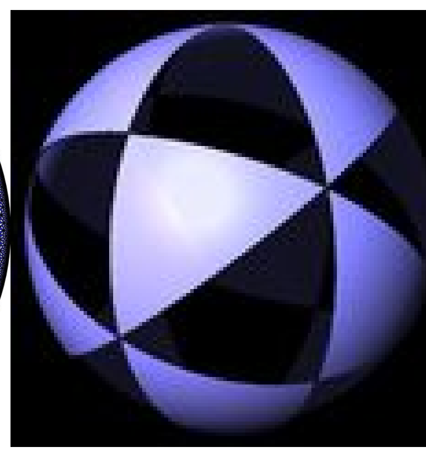
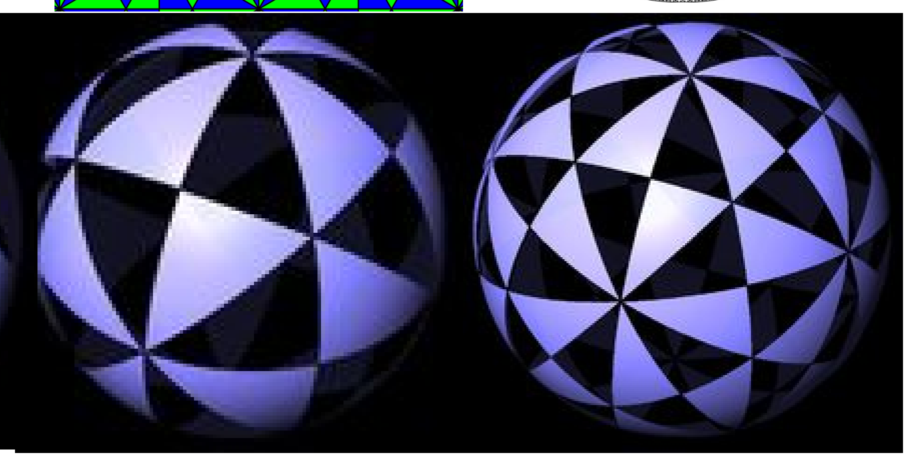
Arrangement of 240 spheres in 8 dimensions

In the E8 sphere packing in 8 dimensions, each sphere touches 240 others. Attempting to draw this in 2 dimensions looks complicated:



Reflection groups

Every symmetry can be obtained by combining several reflections

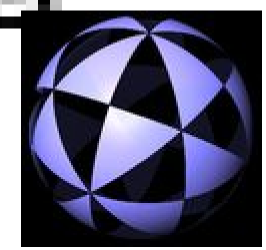








E8 reflection group

The symmetries of a cube has 9 reflections. They are reflections in the planes orthogonal to the following 18 vectors:

(±1, ±1, 0)   Even number of + signs (12 vectors)

(Planes parallel to faces)



(±1, 0, 0) and all vectors of similar “shape” (6 vectors)

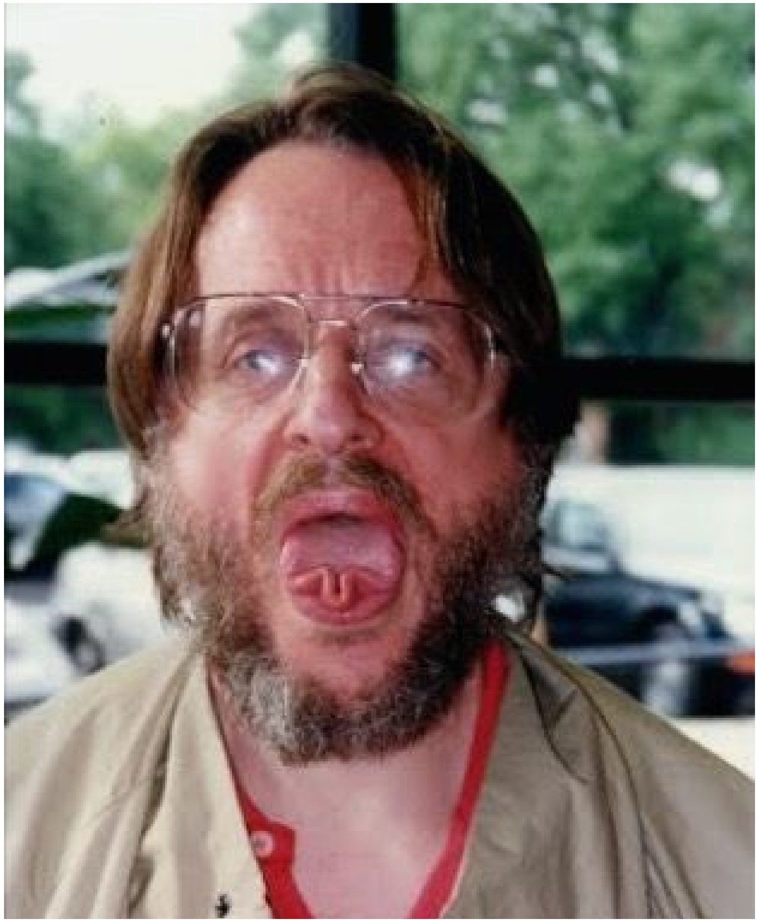
(“Diagonal” reflection planes)

The E8 reflection group has 120 reflections. They are reflections in the hyperplanes orthogonal to the following 240 vectors:

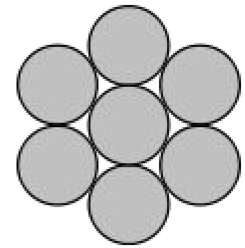
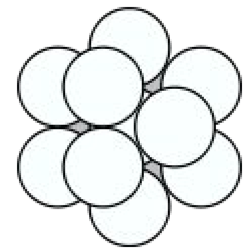
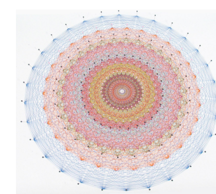
(±½, ±½, ±½, ±½, ±½, ±½, ±½, ±½)   Even number of + signs (128 vectors)

(±1, ±1, 0, 0, 0, 0, 0, 0) and all vectors of similar “shape” (112 vectors)

John Conway



Symmetries of sphere packings

DImension	Spheres touching 1 sphere	Symmetries	
2 Hexagonal	6	12	
3 FCC	12	48	
8 E8	240	696729600	
24 Leech	196560	8315553613086720000	

Conway group

How many cannonballs in pile?

One layer: $1^2 = 1$

Two layers: $1^2 + 2^2 = 5$

Three layers: $1^2 + 2^2 + 3^2 = 14$

N layers: $1^2 + 2^2 + 3^2 + \ldots + N^2 = N(N+1)(2N+1)/6$

24 layers: $1^2 + 2^2 + 3^2 + \ldots + 24^2 = 4900 = 70^2$

The number of cannonballs is a square for 0, 1, or 24 layers

You are not expected to understand this

Leech lattice construction:

Define distance in 26-dimensional space using

$\sqrt{(-a^2+b^2+c^2+ \dots +z^2)}$ (Note minus sign at front)

Put W=(70,0,1,2,3,...,24) so W has length 0 as $0^2 + 1^2 + 2^2 +\dots+ 24^2 = 70^2$.

Leech lattice is (more or less) formed from vectors such that:

*All coordinates are integers n or all are half integers n+½

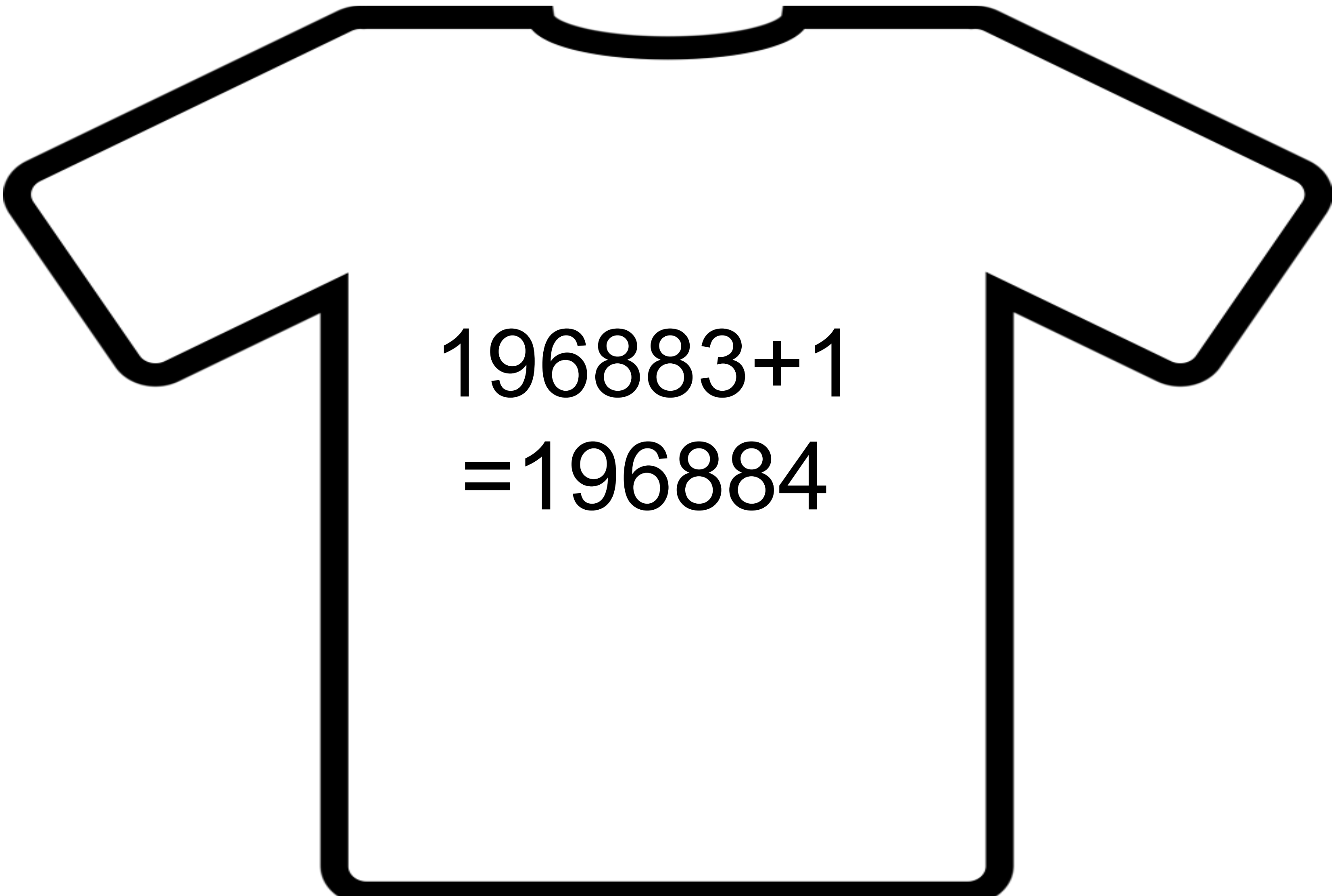
*Sum of coordinates is even

*Vector is orthogonal to W.

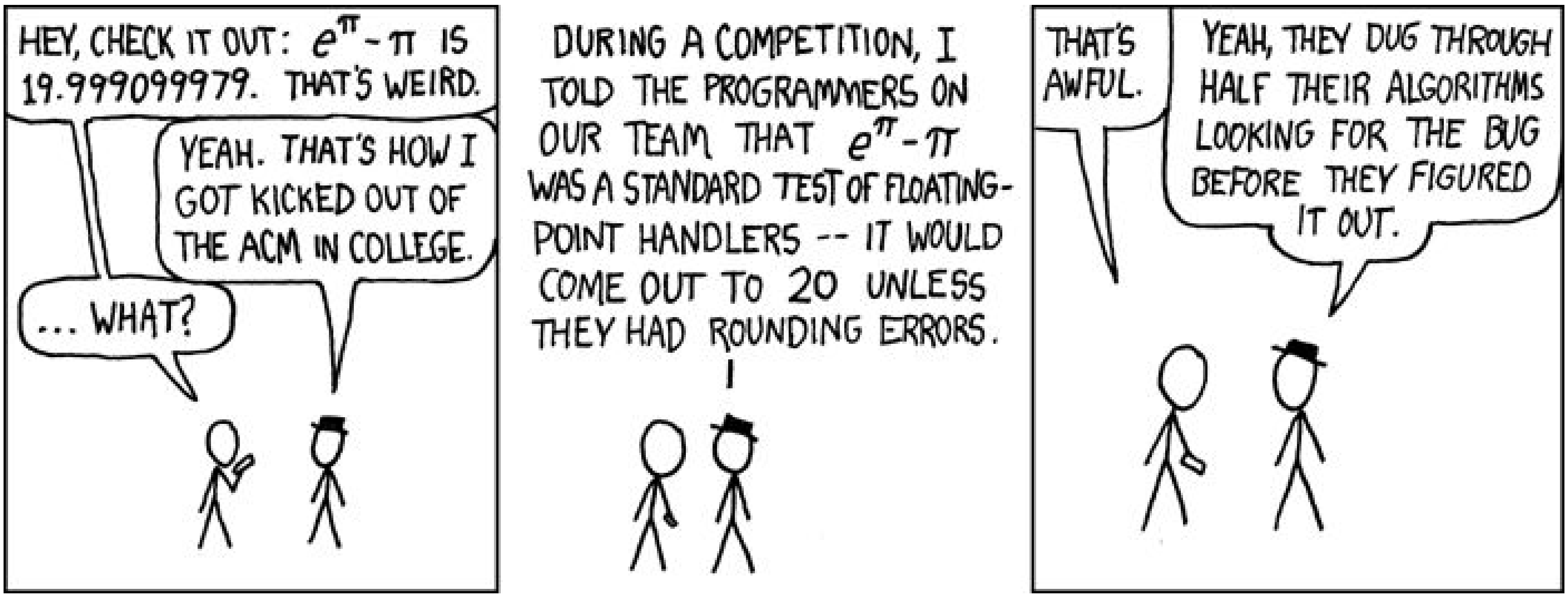
Conway's group is the group of symmetries of the Leech lattice.

This only works in 24 dimensions!

John McKay’s T-shirt



HTTP://XKCD.COM/217/



See http://www.explainxkcd.com/wiki/index.php/217 for explanation

Why are these almost integers?

Ramanujan's second notebook:

$e^{\pi\sqrt{43}}$ = 884736743.999777...

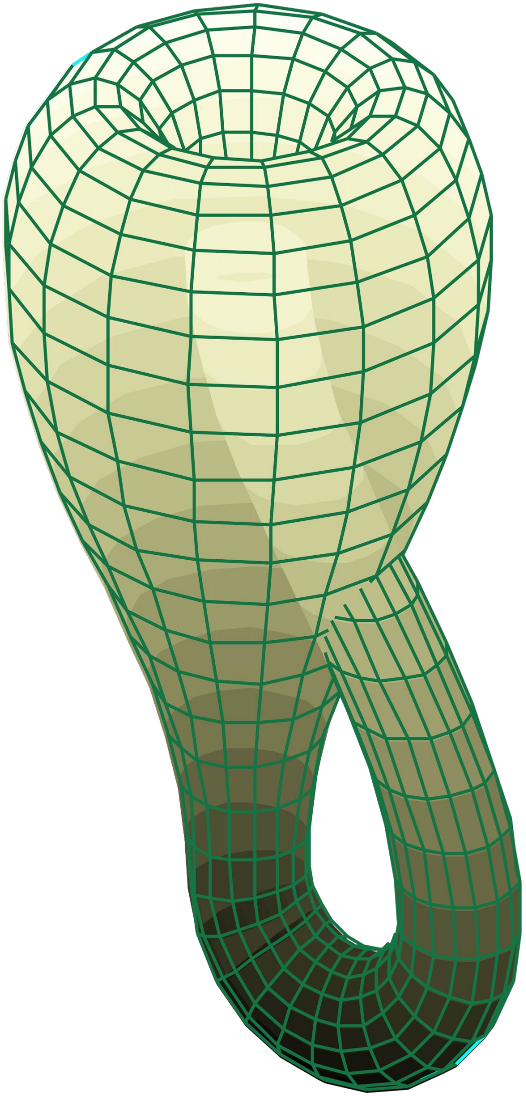
$e^{\pi\sqrt{67}}$ = 147197952743.99999866...

$e^{\pi\sqrt{163}}$ = 262537412640768743.99999999999925...

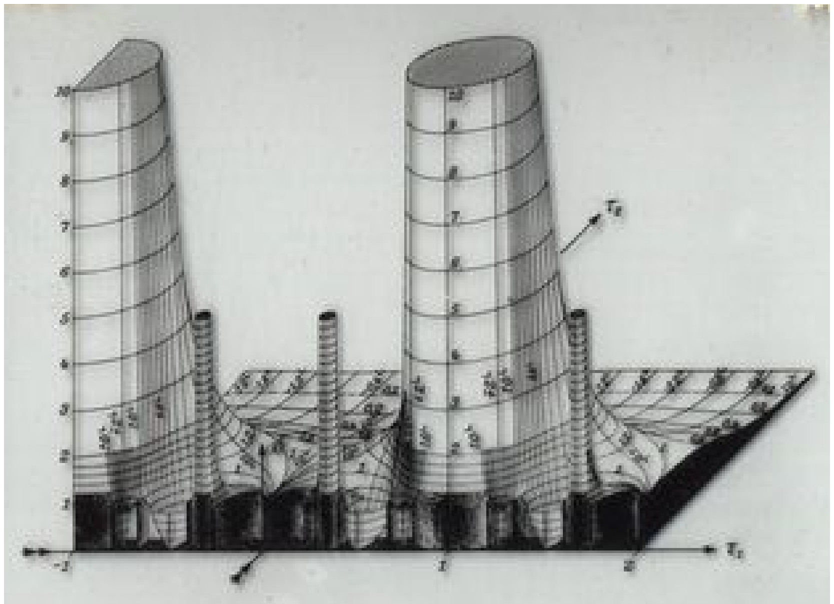
Some are not almost integers: for example,

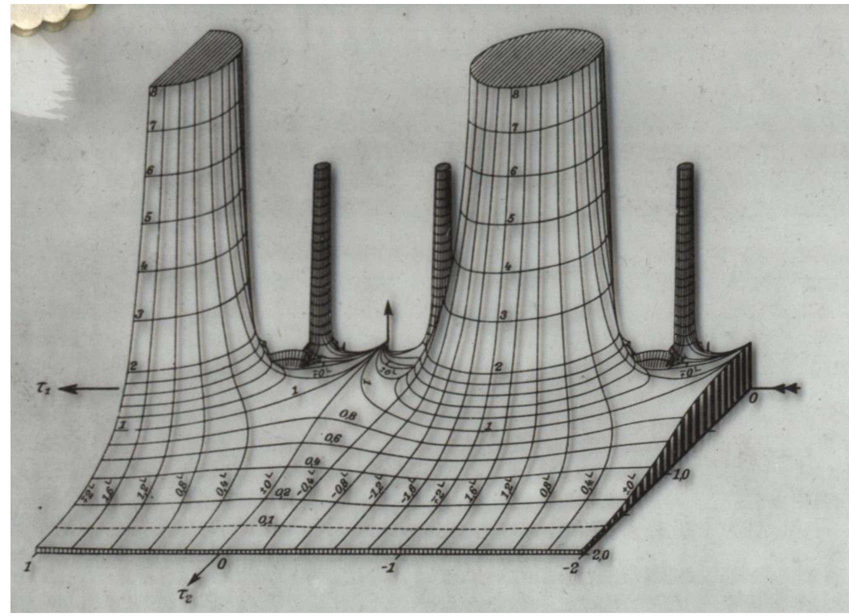
$e^{\pi\sqrt{1}}$ = 23.14069263...

Felix Klein and his bottle



Elliptic modular functions (by Jahnke & Emde)





These show that absolute value of an elliptic modular function on the complex upper half plane.

(These are not pictures of Klein's function, but of a slightly different function.)

Why is $e^{\pi\sqrt{163}}$ almost an integer?

Klein's elliptic modular function is

$$J(q) = 1/q + 744 + 196884\, q + 21493769\, q^2 + \ldots.$$

If $q = -e^{-\pi\sqrt{163}}$ then J(q) is -262537412680768000 = - 640320^3 exactly

(this comes from something called "complex multiplication").

so -262537412680768000 = $-e^{\pi\sqrt{163}}$ + 744 - (something very small).

196884 q looks large, but is in fact tiny because q is so small.

Monster group

Icosahedral group lives in 3 dimensions. 60 symmetries.

Conway group lives in 24 dimensions. 8315553613086720000 symmetries

Fischer - Griess Monster group lives in 196883 dimensions,
808017424794512875886459904961710757005754368000000000 elements

Bernd Fischer

Robert Griess

The monster group in Planetary, book 1, chapter 1

"This is the shape of reality. A theoretical snowflake existing in 196, 833 dimensional space. The snowflake rotates. Each element of the snowflake rotates. Each rotation describes an entirely new universe.

The total number of rotations are equal to the number of atoms making up the earth. "

(Warren Ellis, John Cassady)

Monster character table

	@	@	@	@
		80801742479451287588645	8309562962452852	139511839126
		9904961710757005754368000000000	382355161088000000	336328171520000
		p power	A	A
		p' part	A	A
	ind	1A	2A	2B
χ_1	+	1	1	1
χ_2	+	196883	4371	275
χ_3	+	21296876	91884	-2324
χ_4	+	842609326	1139374	12974
χ_5	+	18538750076	8507516	123004
χ_6	+	19360062527	9362495	-58305
χ_7	+	293553734298	53981850	98970
χ_8	+	3879214937598	337044990	-690690
χ_9	+	36173193327999	1354188159	2864511

Monstrous moonshine

McKay, Thompson, Conway, Norton:

$J(q) = 1/q + 744 + 196884\, q + 21493769\, q^2 + \ldots.$

1	= 1
744	= ????
196884	= 196883+1
21493760	= 21296876 + 196883 + 1
864299970	= 842609326 + 21296876 + 196883 + 196883 + 1 + 1

⇧ ⇧

Coefficients of J   Dimensions in which monster acts on something

Open problem (Umbral moonshine)

Srinivasa
Ramanujan

Ramanujan described mock theta functions in his last

letters to Hardy. One example is:

$H(q) = -1 + 45q + 231q^2 + 770q^3 + 2277q^4 + \cdots$

The Mathieu sporadic group M_{24} of order 244823040

is the symmetries of objects in dimensions

23, **45**, **231**, 252, 483, **770**, 990, 1035, 1265, 1771, 2024, **2277**, ...

Miranda Cheng

Miranda Cheng recently discovered umbral moonshine.

Special case: Why are these numbers the same?

Where to find out more:

Popular books:

M. Ronan "Symmetry and the monster"

M. du Sautoy "Finding Moonshine"

Intermediate:

T. Thompson "From error correcting codes through sphere packings to simple groups"

Advanced:

J. Conway, N. Sloane "Sphere packings, lattices, and groups"