Graph Theory Problems
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1 The Seven Bridges of Königsberg Problem

Königsberg is an ancient city of Prussia, now Kalingrad, Russia. The city was set on both sides of the Pregel River, which also had two islands connected to each other with seven bridges. Two of the seven original bridges were destroyed during WWII, but the story which follows has captured the imagination of mathematicians through present day.

In the Middle Ages, Königsberg was a very important city and trading center with its location strategically positioned on the river. The seven bridges were called Blacksmith's bridge, Connecting Bridge, Green Bridge, Merchant's Bridge, Wooden Bridge, High Bridge, and Honey Bridge. As the story goes, the citizens of Königsberg used to spend Sunday afternoons walking around their beautiful city. While walking, the people of the city decided to create a game for themselves, their goal being to devise a way in which they could walk around the city, crossing each of the seven bridges only once. No one could figure out a route that would allow them to cross each of the bridges only once.[1]

Can you devise a walk through the city that would cross each bridge once, and only once? One caveat: the islands can only be reached by the bridges, and every bridge, once accessed, must be crossed to its other end. The starting and ending points do not have to be the same.

Go ahead and take a few minutes to explore the problem. I won't spoil your fun yet by giving you a chance to try to come up with a solution.

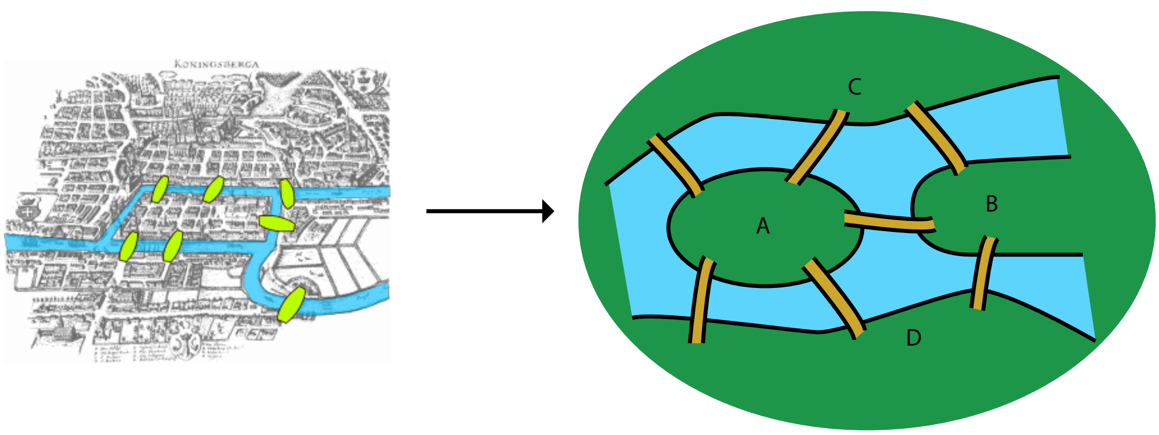


Figure 1: Map of Knigsberg in Euler's time showing the actual layout of the seven bridges, highlighting the river Pregel and the bridges. Photo obtained under GNU license at Wikipedia.

[1] Sources: Wikipedia.org, maa.org

1.1 Euler's Analysis of the Bridge Problem

Luckily for the residents of Königsberg, Leonard Euler lived nearby in St. Petersburg. In a letter written in 1736 to an Italian mathematician, Euler wrote:

> This question is so banal, but seemed to me worthy of attention in that [neither] geometry, nor algebra, nor even the art of counting was sufficient to solve it.[2]

In 1735, Euler presented a paper with the solution to the Königsberg problem, and in doing so he created a branch of mathematics known as graph theory. If you were having trouble thinking of approaches to solving this problem, Euler does, too, in Paragraph 3 of his original paper. But after some exploration and experimentation, Euler decided to ignore the features except for the land masses. He produced a mathematical structure known as a graph:

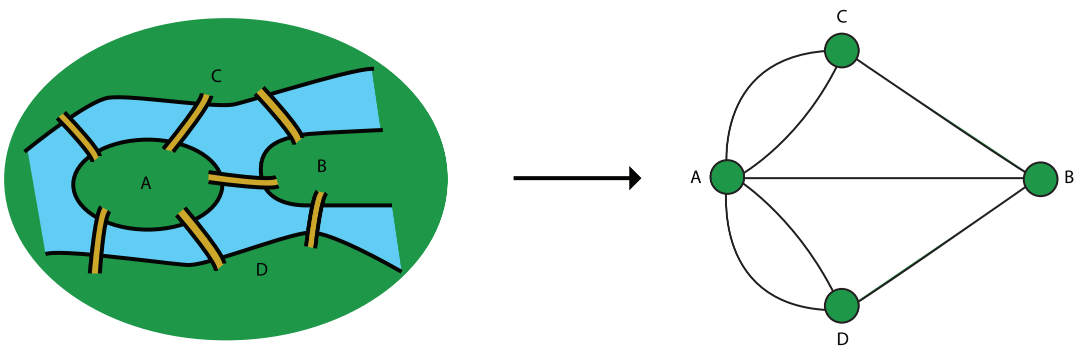


Figure 2: Euler replaces each land mass with a "vertex," or "node."

Definition. Let's define a few definitions in graph theory before we continue:

- A **graph** or **network** is made up of points (**vertices**) connected by lines (**edges**).
- The **degree** or **order** of a vertex is the number of edges that touch it. An **even degree** vertex has an even number of sides connected to it.
- A graph is **traversable** if you can trace the shape without lifting your pen and without going over a side more than once.
- The **size** of a graph is the number of vertices that it has. We often say, "a graph with n vertices."

Problem 1: What is the size of the graph in the figure above? What is the order of vertex B? Is the order of vertex A odd or even?

[2] maa.org

Problem 2: Are the following graphs traversable? Why or why not?

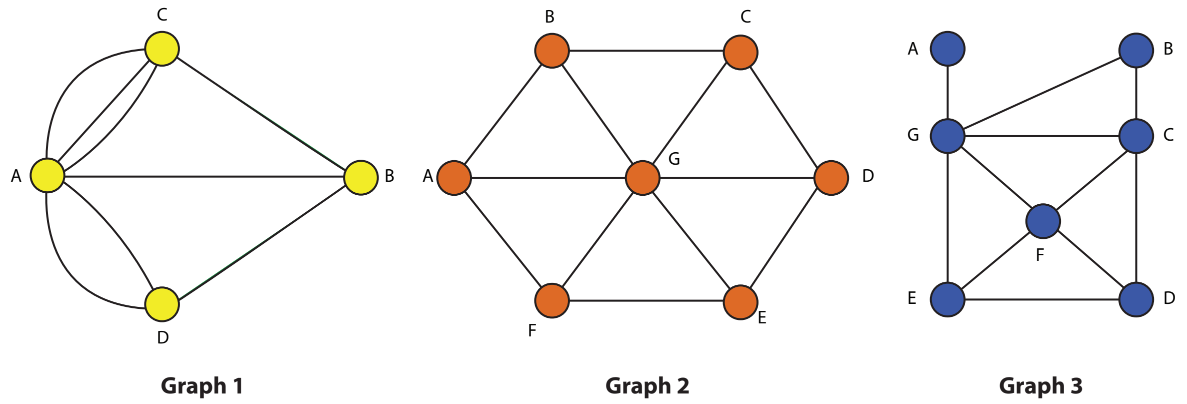
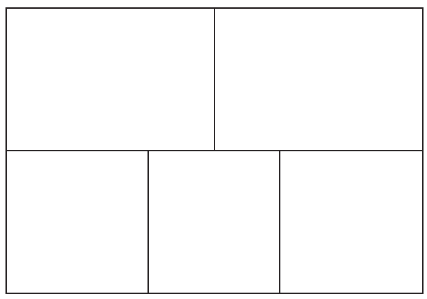


Figure 3: Which of the graphs above are traversable?

Problem 3: Show that there will always be an even number of nodes which have an odd order.

Problems

1. Can you draw a single curve that crosses every line exactly one time?[3]



2. Prove that the sum of the degrees of the vertices of any finite graph is even.

3. In a **simple** graph, two dots can only be connected by one line. Show that every simple graph has two vertices of the same degree.

4. (a) There are 25 students in the class. Is it possible that 6 of them have 9 friends each, 8 of them have 8 friends each, and 11 of them have 7 friends each? (friendships are mutual.) (b) Is it possible to draw 7 line segments on a sheet of paper, so that each of them intersects exactly 3 others?

5. Show that if n people attend a party and some shake hands with others (but not with themselves), then at the end, there are at least two people who have shaken hands with the same number of people.

[3]Sources: Glen Gray, Paul Zeitz, Alexander Remorov

6. Prove that a complete graph with n vertices contains $n(n-1)/2$ edges.

7. A graph is **bipartite** if its vertices can be partitioned into two disjoint sets X and Y so that no two vertices in X are connected by an edge and no two vertices in Y are connected by an edge. Prove that a finite graph is bipartite if and only if it contains no cycles of odd length.

8. Show that if every component of a graph is bipartite, then the graph is bipartite.

9. Prove that if u is a vertex of odd degree in a graph, then there exists a path from u to another vertex v of the graph where v also has odd degree.

10. If the distance $d(u, v)$ between two vertices u and v that can be connected by a path in a graph is defined to be the length of the shortest path connecting them, then prove that the distance function satisfies the triangle inequality: $d(u, v) + d(v, w) \geq d(u, w)$.

11. Consider the sequence 01110100 as being arranged in a circular pattern. Notice that every one of the eight possible binary triples: 000, 001, 011, . . . , 111 appear exactly once in the circular list. Can you construct a similar list of length 16 where all the four binary digit patterns appear exactly once each? Of length 32 where all five binary digit patterns appear exactly once?

12. An n-cube is a cube in n dimensions. A cube in one dimension is a line segment; in two dimensions, it's a square, in three, a normal cube, and in general, to go to the next dimension, a copy of the cube is made and all corresponding vertices are connected. If we consider the cube to be composed of the vertices and edges only, show that every n-cube can be modeled as a graph that visits each vertex exactly once.

13. A **tree** is a connected graph with no cycles, and a **cycle** is a path which begins and ends on the same vertex. How many edges does a tree have? Show that a tree with n vertices has exactly $n-1$ edges.

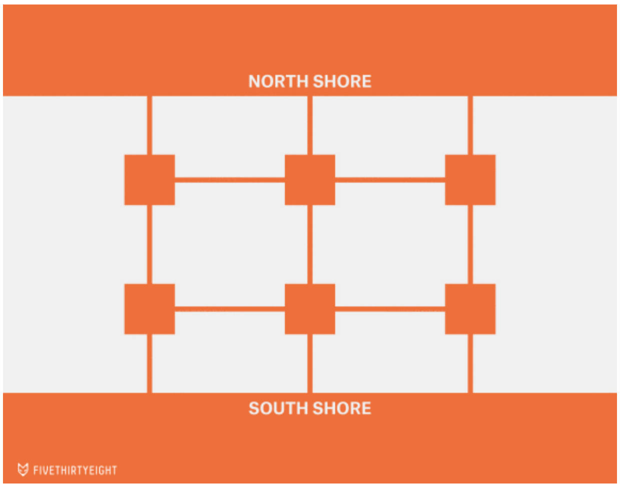
14. If u and v are two vertices of a tree, show that there is a unique path connecting them.

15. Show that any tree with at least two vertices is bipartite.

2 Challenge of the Day

This problem was posed on Five Thirty-Eight's blog, *The Weekly Riddler*, by Chris Scrambler. You can find the solution on Five Thirty Eight, but you'll have more fun if you think about the problem first!

You're on the north shore of a river, and want to cross to the south, via a series of 13 bridges and six islands, which you can see in the diagram below. But, as you approach the water, night falls, a bad storm rolls in, and you're forced to wait until morning to try to cross. Overnight, the storm could render some of the bridges unusable – it has a 50 percent chance of knocking out each of the bridges. (The chance is independent for each bridge.)



What's the probability you will be able to cross the river in the morning? (You have no boat, can't swim, can't fix the bridges, etc. No tricks!)