Circle Geometry

Inspired by Chapter 11 of *A Decade of the Berkeley Math Circle, Volume 1.*

September 15, 2015

Introduction. Before we discuss circle geometry, let's review a simple but extremely useful theorem about triangles.

Theorem 1 (The Exterior Angle Theorem). *An exterior angle is the sum of the remote interior angles, and hence it is larger than each of them.*

To put this more precisely, given triangle ABC, let point X lie on line BA such that B is between X and A (as shown in figure 1). Then

(a) $\angle XBC = \angle BCA + \angle CAB$

(b) $\angle XBC$ is larger than either of $\angle BCA$ or $\angle CAB$.

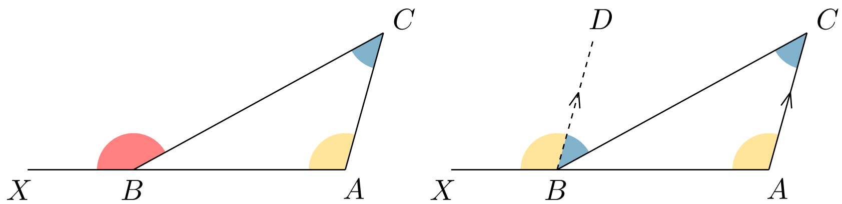
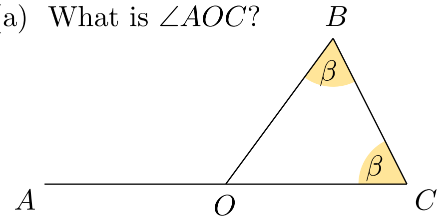


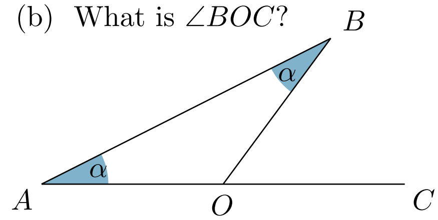
Figure 1: Exterior angle XBC is the sum of remote interior angles BCA and CAB.

Exercise 1. Here is some easy practice with the exterior angle theorem.

(a) What is $\angle AOC$?



(b) What is $\angle BOC$?



Theorem 2 (Thales' Theorem). *If A, B, and C are points on a circle, and AC is a diameter of the circle, then $\angle ABC$ is a right angle.*

Thale's theorem is named for Thales of Miletus, a Greek philosopher and mathematician. Born circa 624 BC, Thales is sometimes called the first Greek mathematician. He predates Pythagoras by decades and Euclid by centuries.

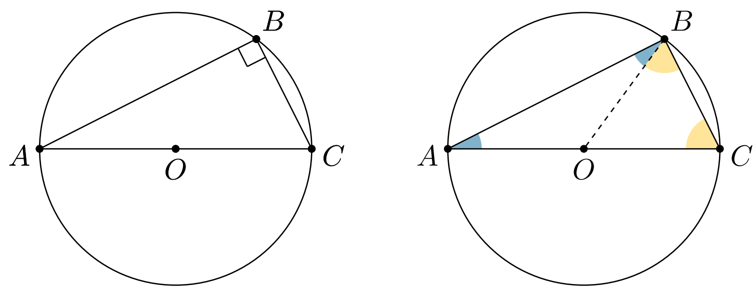


Figure 2: (a) Thales' theorem, and (b) a strategy for its proof.

Proof of Thales' Theorem: Since $OA = OB = OC$, triangles AOB and AOC are isosceles. The isosceles triangle theorem says that the base angles of an isosceles triangle are congruent, which means $\angle OAB = \angle OBA$ and $\angle OBC = \angle OCB$. In figure 2b, isosceles triangles AOB and BOC are shown with their congruent base angles indicated in blue and yellow, respectively.

Let $\alpha = \angle OAB = \angle OBA$ and $\beta = \angle OBC = \angle OCB$. By the exterior angle theorem, $\angle AOB = 2\beta$ and $\angle BOC = 2\alpha$. Since $\angle AOC$ is a straight angle, $\angle AOC = 180°$. Therefore, by angle addition,

$$\begin{aligned} & \angle AOB + \angle BOC = 180^\circ \\ \iff & \quad 2\beta + 2\alpha = 180^\circ \\ \iff & \quad \beta + \alpha = 90^\circ \end{aligned}$$

Notice that $\angle ABC = \alpha + \beta$, so we have that $\angle ABC = 90°$. □

Central and Inscribed Angles. In a circle, the angle formed by two radii is called a *central angle*. As the name suggests, the vertex of a central angle is located at the center of the circle. The figure below shows three different central angles (marked in yellow) that could be described by the name $\angle AOB$. Each central angle cuts off *arc* AB of circle O (shown below in red). We define the measure of an arc as the size of its central angle.

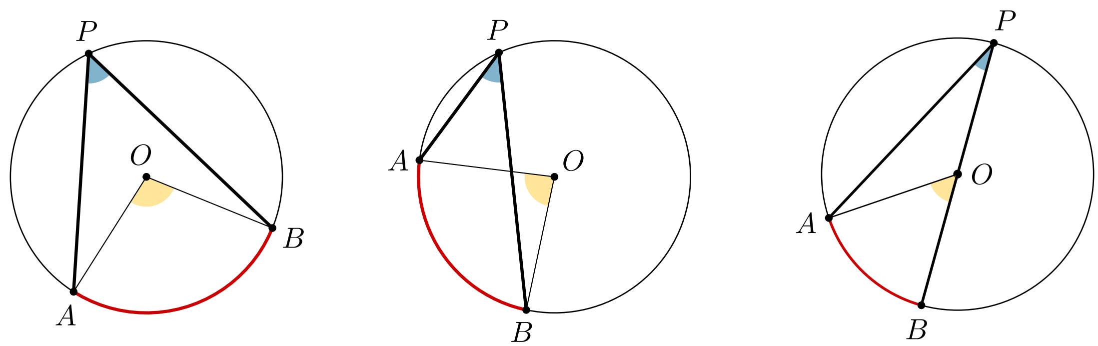


Figure 3: An inscribed angle APB and its corresponding central angle AOB (3 cases).

In a circle, an angle formed by two chords joined at one endpoint is called an *inscribed angle*. The figure above shows three instances of central angle AOB and inscribed angle APB intercepting the same arc. We say that inscribed angle APB and central angle AOB *subtend* the same arc, or that central angle AOB *corresponds* with inscribed angle APB.

Theorem 3 (**The Inscribed Angle Theorem).** *An inscribed angle is half the measure of the corresponding central angle.*

PARTIAL PROOF: The easiest case occurs is when one side of inscribed angle APB is a diameter (see figure 4, left). In this case, $\angle AOB$ is an exterior angle to isosceles triangle AOP. As we saw earlier, the exterior angle theorem tells us that $\angle AOB = 2(\angle APB)$, or equivalently, that $\angle APB$ is *half* of $\angle AOB$.

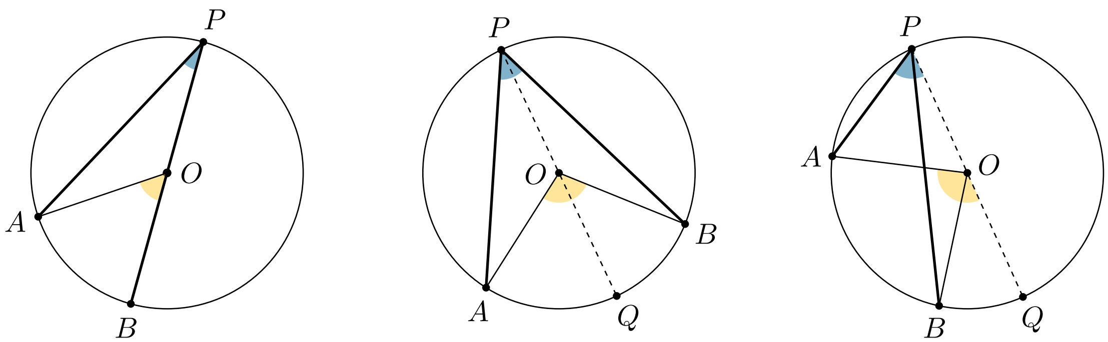


Figure 4: Inscribed angle APB is half of its corresponding central angle AOB.

If neither side of $\angle APB$ is a diameter of the circle, then we can start by drawing diameter PQ. Figure 4 (middle and right) shows the two possible cases. We'll just consider the middle case. Observe that $\angle AOQ$ is exterior to isosceles triangle AOP, and that $\angle BOQ$ is exterior to isosceles triangle BOP. As in our first case, we have $\angle APQ = \frac{1}{2}\angle AOQ$ and $\angle QPB = \frac{1}{2}\angle QOB$. Thus,

$$\begin{aligned} \angle APB &= \angle APQ + \angle QPB \\ &= \tfrac{1}{2}\angle AOQ + \tfrac{1}{2}\angle QOB = \tfrac{1}{2}\angle AOB. \end{aligned} \qquad \square$$

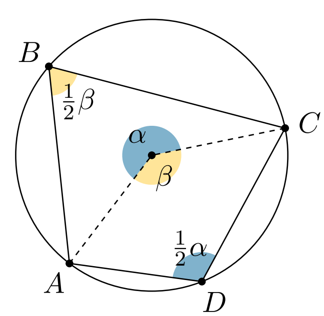
Corollary 1. *Inscribed angles that intercept the same arc are congruent.*

Exercise 2. Let AB and CD be two chords of the same circle. Then $AB = CD$ if and only if the corresponding inscribed angles are equal.

Cyclic Quadrilaterals. A quadrilateral whose vertices all lie on the same circle is called *cyclic*.

Theorem 4. *Quadrilateral $ABCD$ is cyclic if and only if $\angle ABC + \angle ADC = 180°$.*

Proof: ($\Longrightarrow$ only) Suppose quadrilateral $ABCD$ is inscribed in a circle, as shown at right. By the inscribed angle theorem, $\angle ABC = \frac{1}{2}\alpha$ and $\angle ADC = \frac{1}{2}\beta$.

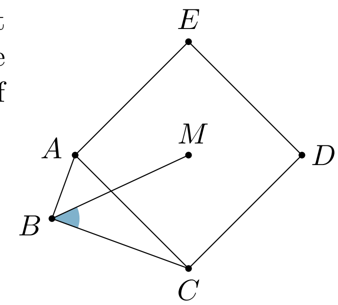


Since $\alpha + \beta = 360°$,

$$\angle ABC + \angle ADC = \tfrac{1}{2}\alpha + \tfrac{1}{2}\beta = 180°.$$

If you're curious about the ($\Longleftarrow$) direction of the proof, see page 229 of *A Decade of the Berkeley Math Circle, Volume 1.*

Problem 1 (BAMO 2002). In $\triangle ABC$, $\angle B$ is a right angle. Let $ACDE$ be a square drawn exterior to $\triangle ABC$, as shown in the figure on the right. If M is the center of this square, find the measure of $\angle MBC$.



Theorem 5 (Ptolemy's Theorem.). *If $ABCD$ is a cyclic quadrilateral, then*

$$AC \cdot BD = AB \cdot CD + AD \cdot BC.$$

PROOF: Figure 5, left, shows cyclic quadrilateral $ABCD$ and its diagonals AC and BD. Inscribed angles BAC and BDC are equal by Corollary 1, since they both intercept arc BC. Likewise, inscribed angles ADB and ACB are equal since they both intercept arc AB. Let P be the point in diagonal AC that makes $\angle ABP = \angle DBC$. Since $\angle ABP + \angle PBC = \angle ABD + \angle DBC$ (both sums equal $\angle ABC$), subtracting the equation $\angle ABP = \angle DBC$ yields $\angle PBC = \angle ABD$.

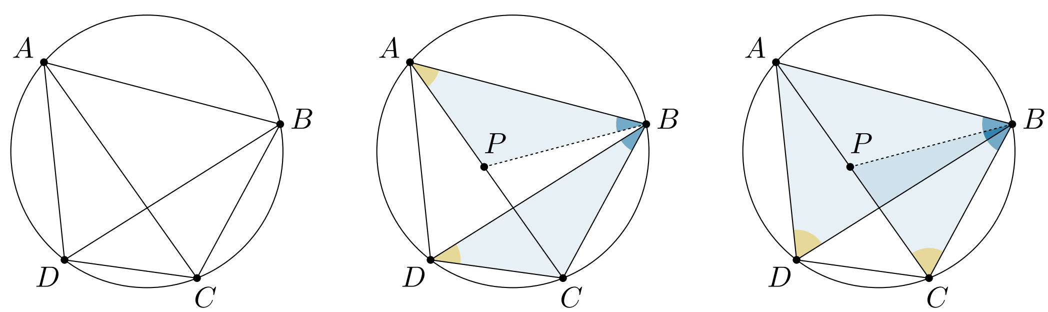


Figure 5: The proof of Ptolemy's theorem uses similar triangles.

Since $\angle BAC = \angle BDC$ and $\angle ABP = \angle DBC$, $\triangle ABP$ and $\triangle DBC$ are similar (see figure 5, middle). Thus, $AP : AB = CD : BD$, and so

$$AP \cdot BD = AB \cdot CD. \tag{1}$$

Since $\angle ADB = \angle ACB$ and $\angle ABD = \angle PBC$, $\triangle ABD$ and $\triangle PBC$ are similar (see figure 5, right). Thus, $PC : BC = AD : BD$, and so

$$PC \cdot BD = AD \cdot BC. \tag{2}$$

Adding (1) and (2), we get

$$\begin{aligned} AP \cdot BD + PC \cdot BD &= AB \cdot CD + AD \cdot BC \\ \Longrightarrow\ (AP + PC) \cdot BD &= AB \cdot CD + AD \cdot BC \\ \Longrightarrow\ AC \cdot BD &= AB \cdot CD + AD \cdot BC \end{aligned}$$

□

Exercise 3. A regular pentagon has sides of length 1. How long are its diagonals?

Problem 2 (1991 AIME Problem 14). A hexagon is inscribed in a circle. Five of the sides have length 81 and the sixth, denoted by $\overline{AB}$, has length 31. Find the sum of the lengths of the three diagonals that can be drawn from A.