HOW TO WIN ALL THE TIME
(AND NOT BY CHEATING)

BMC
Session 1
9.1.2015

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(AND NOT BY CHEATING)

GAME THEORY: STRATEGIC INVARIANTS

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BIG IDEA:

WHEN YOU PLAY WITH A NEW IDEA IN MATH, YOU WANT TO FIGURE OUT HOW TO PURPOSEFULLY CHANGE THE OUTCOME.

THEN YOU LOOK FOR PATTERNS: WHAT ***CHANGES***, WHAT **STAYS THE SAME**? WHEN SOMETHING STAYS THE SAME, IT'S CALLED

AN *INVARIANT*.

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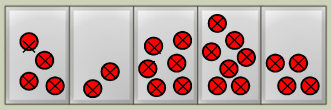
NIM

Nim is cool because it's part of A LOT of games.

Rules:
There are many stones distributed among a bunch of squares.
On your turn, you can pick ONE square and remove any number of stones The person to take the last stone from the game wins.

Player 1: __ME__ Player 2: __ALSO ME__



I WIN! ;D

PLAYER 1 WINS!

WILL PLAYER 1 ALWAYS WIN THIS GAME?

TWO SQUARE NIM 10-10

Goal:
Be the player who takes the last stone in the whole game.

Win All Games – pg1

Two-Square Nim 10-10

Put **10 tokens** in square A and **10 tokens** in square B. On your turn, you can take any number of tokens from **one and only one** of the two squares. The person to take the **last** token in the entire game wins.

1) Does player 1 or player 2 have the advantage in this game? (circle one) **P1** **P2**

Win All Games – pg2

Two-Square Nim 7-13

Put **7 tokens** in square A and **13 tokens** in square B and play with the same rules as before.

2) Does player 1 or player 2 have the advantage in this game? (circle one) **P1** **P2**
3) What strategy can they use to win every game?
4) Which Two-Square Nim games (X-Y) that can be won with this strategy?

TWO SQUARE NIM
7-7-6

3) Does player 1 or player 2 have the advantage in this game? (circle one) **P1** **P2**
6) What strategy can they use to win every game?

Goal:
Be the player who takes the last stone in the whole game.

Win All Games – pg4

Three-Square Nim (?-?-?)

Design your own 3-square Nim game by deciding on the starting distribution of tokens.

7) What distribution did you choose? _____ - _____ - _____

8) Does player 1 or player 2 have the advantage in your game? (circle 1) **P1** **P2**
9) What strategy can they use to win every game?

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10) Convince yourself that the chart below includes every game with 20 total stones.
11) Figure out which player has the advantage for as many games as you can.
12) What pattern(s) do you see and what strategies do the players need to know?

COLOR KEY ☐ = Player 1 Wins ☐ = Player 2 Wins

20-0-0						
19-1-0						
18-2-0	18-1-1					
17-3-0	17-2-1					
16-4-0	16-3-1	16-2-2				
15-5-0	15-4-1	15-3-2				
14-6-0	14-5-1	14-4-2	14-3-3			
13-7-0	13-6-1	13-5-2	13-4-3			
12-8-0	12-7-1	12-6-2	12-5-3	12-4-4		
11-9-0	11-8-1	11-7-2	11-6-3	11-5-4		
10-10-0	10-9-1	10-8-2	10-7-3	10-6-4	10-5-5	
		9-9-2	9-8-3	9-7-4	9-6-5	
				8-8-4	8-7-5	8-6-6
						7-7-6

Name: _____

Notes – BMC Math Circle – How to Win Every Game – 9/1/15

Binary – Base 2

What if you didn't want to teach 10 symbols (0, 1, 2, 3, 4, 5, 6, 7, 8, 9) in school, but instead, only 2 (0, and 1)? Make a list, in increasing order, of all of the numbers that only contain the digits 0 and 1?

0: 0	8:	16:	SPICY CHALLENGE:
1: 1	9:	17:	What do you think the
2: 10	10:	18:	100th such number will be?
3: 11	11:	18:	
4: 100	12:	20:	…
5:	13:	21:	100:
6:	14:	22:	101:
7:	15:	23:	102:

Can we explain these numbers in such a way that 100 *means* 4, 1000 *means* 8, and 10011 *means* 19?

BINARY

THIS IS BINARY:

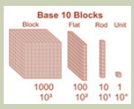
1 = 0001
2 = 0010
3 = 0011
4 = 0100
5 = 0101
6 = 0110
7 = 0111
8 = 1000
9 = 1001
10 = 1010
11 = 1011
Etc.

What's going on?

In short, we're using a base 2 system to write numbers instead of base 10.

Base 10
3254 means
"three thousands, two hundreds, five tens, and four ones"

Base
1101 means
"one 8, one 4
no 2s and one 1"
AKA, it means 13



Powers

	0	1	2	3	4	5	6	7
Base 10								
Base 2								

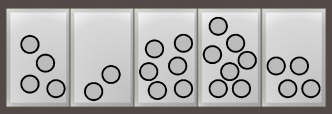
# in base 10	Same # in base 2 (0 and 1 are the only digits) 128s	64s	16s	8s	4s	2s	1s
2	___	___	___	___	___	___	___
20	___	___	___	___	___	___	___
3	___	___	___	___	___	___	___
8	___	___	___	___	___	___	___
16	___	___	___	___	___	___	___
23	___	___	___	___	___	___	___
64	___	___	___	___	___	___	___
63	___	___	___	___	___	___	___
127	___	___	___	___	___	___	___

20 – 10100 0 – 00000 0 – 00000				
19 – 10011 1 – 00001 0 – 00000				
18 – 10010 2 – 00010 0 – 00000	18 – 10010 1 – 00001 1 – 00001			
17 – 10001 3 – 00011 0 – 00000	17 – 10001 2 – 00010 1 – 00001			
16 – 10000 4 – 00100 0 – 00000	16 – 10000 3 – 00011 1 – 00001	16 – 10000 2 – 00010 2 – 00010		
15 – 01111 5 – 00101 0 – 00000	15 – 01111 4 – 00100 1 – 00001	15 – 01111 3 – 00011 2 – 00010		
14 – 01110 6 – 00110 0 – 00000	14 – 01110 5 – 00101 1 – 00001	14 – 01110 4 – 00100 2 – 00010	14 – 01110 3 – 00011 3 – 00011	

NIM IN BINARY

Can we find a pattern to which games are won by player 2?

WHAT'S THE PATTERN?



0100 0010 0110 0111 0100

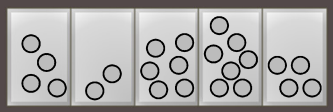
Step 1: Write out the number of stones in each box in binary. (= ????)

Step 2: Put these numbers in a column and figure out if the number of 1's in each column is even or odd. If it's even, write 0 at the bottom, if it's odd write 1

0100
0010
0110
0111
0100
0011

This is called the Nim-Sum

WHAT'S THE PATTERN?



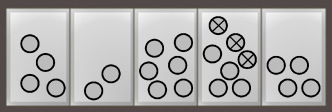
0100 0010 0110 0111 0100

Step 3: If the Nim-Sum Is ALL ZEROS, choose to be player 2. If Nim-Sum is NOT ZERO, choose to be player 1.

Play: On your move, make the Nim-Sum ZERO. Eventually, it will be 0 because there are no stones left and you will win!

0100
0010
0110
0111
0100
0011

WHAT'S THE PATTERN?



0100 0010 0110 ~~0111~~ 0100
0100

Play: On your move, make the Nim-Sum ZERO.

0100
0010
0110
0100
0100
0000

1) Choose to be player…

Choose to be player 1, because the Nim-sum is NON-ZERO

2) What can you do to make the Nim-Sum 0?…

Find a box that has 11 at the end and take out 3 stones.

WHAT'S THE PATTERN?

0100 0010 ~~0110~~ 0100 0100
0000

Play: On your move, make the Nim-Sum ZERO.

0100
0010
0000
0100
0100
0110

1) Choose to be player…

Choose to be player 1, because the Nim-sum is NON-ZERO

2) What can you do to make the Nim-Sum 0?…

Find a box that has 11 at the end and take out 3 stones.

Then your opponent moves and will necessarily

Make the nim-sum non-zero again (PROVE IT!)

3) REPEAT

STEP 2: Make the Nim-Sum 0

WHAT'S THE PATTERN?

~~0100~~ 0010 0000 0100 0100
0010

Play: On your move, make the Nim-Sum ZERO.

0010
0010
0000
0100
0100
0000

1) Choose to be player…

Choose to be player 1, because the Nim-sum is NON-ZERO

2) What can you do to make the Nim-Sum 0?…

Find a box that has 11 at the end and take out 3 stones.

Then your opponent moves and will necessarily

Make the nim-sum non-zero again (PROVE IT!)

3) REPEAT

STEP 2: Make the Nim-Sum 0

Sometimes it's tricky, but you can always get it back to zero again (PROVE IT!)

FINISH THE GAME

0010 0010 0100 0101 0001

YOU WIN!

Play... your move, make the Ni... Sum ZERO.

0010
0010
0000
0001
0100
0101

1) Choose to be player...
Choo... to be ...ver... because t... Nim-...m is NO...ERO.
...t c... you do... make the Nim-...m 0?...
Find a box that has ...d at the end and take out 3 stones.
Then your opponent moves and will necessarily Make the nim-sum non-zero again (PROVE IT!)
3) REPEAT STEP 2: Make the Nim-Sum 0 sometimes it's tricky, but you can always get it back to zero again (PROVE IT!)

Solving Nim

The Nim-Sum of a game is 0 if: Considering the number of stones in each pile written in binary, in every place value, there are an even number of 1's

The games that can be won by player 1 are: Games in which the initial Nim-Sum is non-zero.

The games that can be won by player 2 are: Games in which the initial Nim-Sum is 0.

An Invariant is…

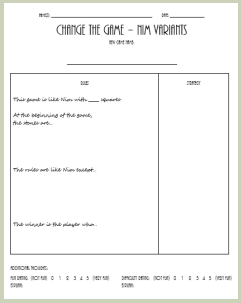
Intuitive definition (describe it in simple word):

Something that stays the same.

Rigorous definition (the precise mathematical definition):

A property, held by a class of **mathematical** objects, which remains unchanged when transformations of a certain type are applied to the objects.

FOR NEXT WEEK:



CHANGE THE GAME – NIM VARIANTS

Sneak preview of next week…
"Magic with a Half-Deck" — Base 3

What if we allow 3 numbers in our system: 0, 1, and 2

0:	5:	10:	15:	20:	25:	Can you make
1:	6:	11:	16:	21:	26:	the same kind
2:	7:	12:	17:	22:	27:	of reasoning
3:	8:	13:	18:	23:	28:	here as we did
4:	9:	14:	19:	24:	29:	for base 2?

CONTACT INFO

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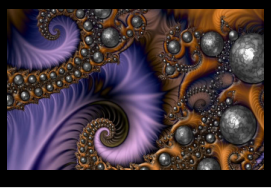
If you want the presentation, just email me!
Thanks! And have a great week!



MY FAVORITE QUOTES

1) Cantor: *"In mathematics the art of proposing a question must be held of higher value than solving it."*

2) Howard Thurman: *"Don't ask what the world needs. Ask what makes you come alive, and go do it. Because what the world needs is people who have come alive."*

3) Richard Feynman: *"Nobody ever figures out what life is all about, and it doesn't matter. Explore the world. Nearly everything is really interesting if you go into it deeply enough."*