Triangles, Altitudes, and Area

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***Note:** This BMC session is inspired from a variety of sources, including AwesomeMath, Areteem Math Zoom, *A Decade of Math Circles, Vol. 1* by Zvezdelina Stankova and Tom Rike, and Bulgarian mathematician and educator Professor Georgi Paskalev. Mostly, we are very grateful to the instructors, mathematicians, and mentors who have inspired great teaching over the years.

1 Introduction

Geometry is the mathematics of shape, and it is best understood with the help of pictures. *Construction* means to accurately draw a picture with the help of a straightedge and compass. Sometimes you want to use a right triangle tool, patty paper, or a protractor. What can we make with our straight lines and circles? In this session, we apply simple logical arguments to the simplest of assumptions in order to produce beautiful results.

Let's begin with a warm-up problem:[1]

Problem 1: Using the figure below, what point P on the upper line should be chosen so that the triangle formed by X, Y, and P has the greatest area?

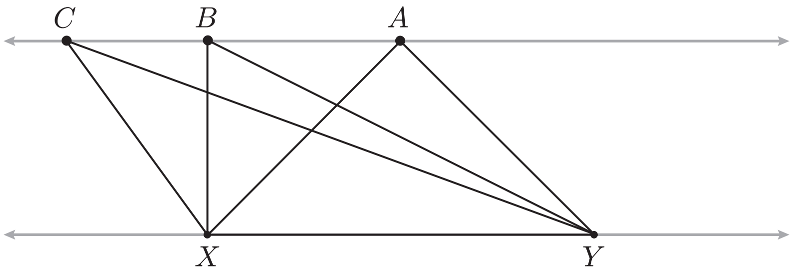


Figure 1: Which point on the upper line results in a triangle with the greatest area?

[1]The question and figure on this page are from *The Magic of Math* by Arthur Benjamin. Illustration by Natalya St. Clair.

Definition. **The Distance from a Point to a Line.** In geometry, the shortest distance from a point to a line is the **perpendicular distance.**

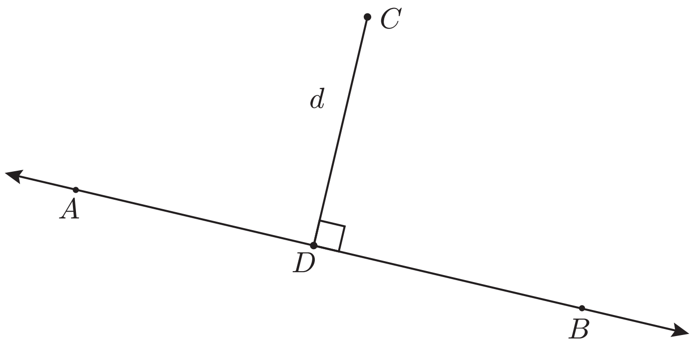


Figure 2: $\overline{CD}$ is the perpendicular dropped from point D to line AC. Notice that $\overline{CD} \perp$ line AB.

Remark. Remember that CD is the *length* of line segment $\overline{CD}$. CD is a number and a distance from C to D!

Exercise 1: Use your straightedge to drop a perpendicular line passing through a line and a point on a sheet of paper. How can you be certain that your line is really the shortest distance? Try flipping the paper upside down to draw a few more perpendicular lines.

Exercise 2: Using the previous problem, draw the three perpendiculars from the vertices of the triangles in the figures below. Can you explain what the height of the triangle is using this definition?[2]

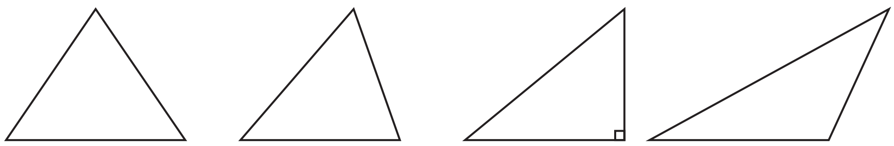


Figure 3: Equilateral triangle, acute triangle, right triangle, and obtuse triangle. What are the types of triangles according to their angles?

Using only a straightedge and a right triangle tool, show that is is possible to construct the altitudes of the triangle. Give an *algorithm* for each construction and prove that it does what is is supposed to do.

Construction 1: Given a segment BC, find the distances from a given point a to BC.

Construction 2: Draw $\triangle ABC$ and its three altitudes.

[2]This figure is from *The Magic of Math* by Arthur Benjamin. Illustration by Natalya St. Clair.

An alternate but very intuitive way to approach constructions 1–2 uses paper folding. For example, if you are given a segment BC drawn on paper, you can *fold* the perpendicular segment on patty paper.

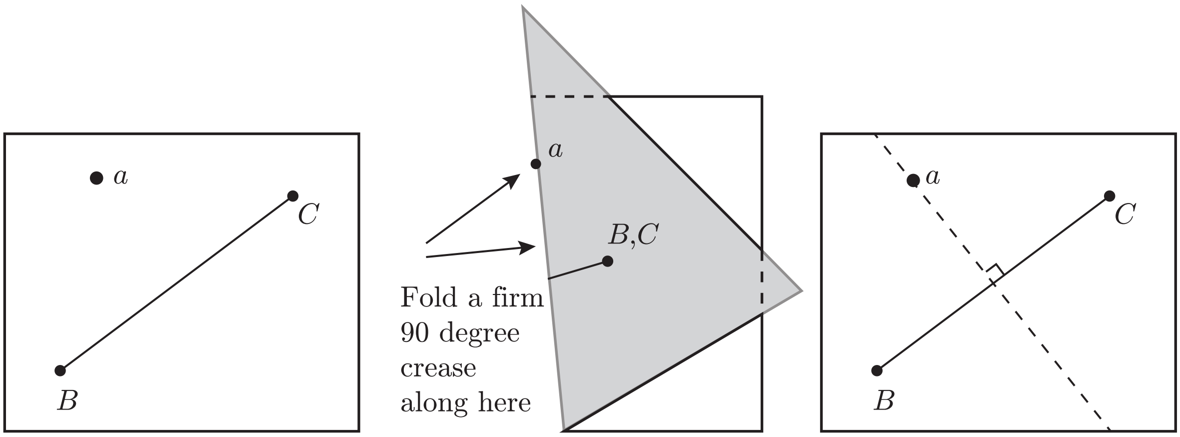


Figure 4: Folding the perpendicular distance from point a to line segment BC.

Definition. The *perimeter* of a polygon is the sum of the lengths of its sides. We define the *area* of a 1-by-1 square (the *unit square*) to have area 1. When b and h are positive integers, like in the figure below, we can break up the region into bh 1-by-1 squares, so its area is bh. In general, for any rectangle with base b and height h, (where b and h are positive, but not necessarily integers) we define its area to be bh.[3]

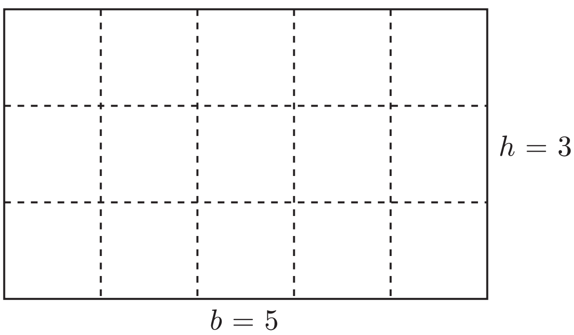


Figure 5: A rectangle with base b and height h has perimeter $2b + 2h$ and area bh.

Speaking of area, let's go back to the triangle problem. Starting with the area of a rectangle, it is possible to derive the area of just about any geometrical figure. First and foremost, we define the area of the triangle:

Definition. A triangle with base b and height h has area $\frac{1}{2}bh$.

[3]This figure and the one on the next page are from *The Magic of Math* by Arthur Benjamin. Illustration by Natalya St. Clair.

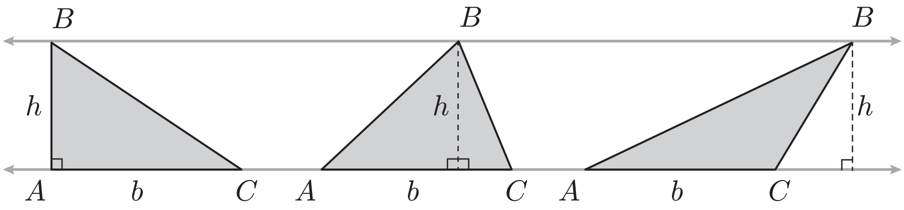


Figure 6: The area of a triangle with base b and height h is $\frac{1}{2}bh$. This is true, regardless of whether the triangle is right-angled, acute, or obtuse.

Problems

Remark. **Notation:** We shall use $[ABC]$ to denote the area of triangle ABC, $[XYZW]$ to denote the area of the quadrilateral $XYZW$.

Formulas for areas (should be memorized): triangle, rectangle, square.[4]

Shape	**Perimeter**	**Area**
Triangle	$a+b+c$	$\frac{1}{2}bh$
Rectangle	$2a+2b$	ba
Square	$4a$	$a \cdot a = a^2$

Figure 7: Various area and perimeter formulas.

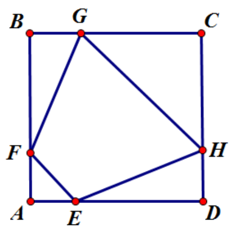
The areas of triangles (or parallelograms) with equal bases and equal altitudes (heights) are equal.

1. Prove the Pythagorean Theorem using areas.

2. If $AB \parallel CD$, can we conclude that $[ABC] = [ABD]$?

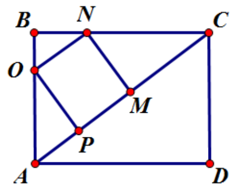
3. (2002 AMC 12A #22) Triangle ABC is a right triangle with $\angle ACB$ as its right angle, $m\angle ABC = 60°$, and $AB = 10$. Let P be randomly chosen inside $\triangle ABC$, and extend $\overline{BP}$ to meet $\overline{AC}$ at D. What is the probability that $BD > 5\sqrt{2}$?

4. Given that $ABCD$ is a square, $AF = BG = 5$, and $BF = CH = DE = 12$, compute the area of $EFGH$.



[4]The figures in the problems are from Areteem Math Zoom Academy, 2014.

5. In the figure $ABCD$ is a rectangle, $AO = 15, BO = 6, AD = 28$. Find the area of rectangle $MNOP$.

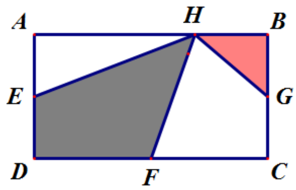


6. If the side length of an equilateral triangle is 5, what is the area? What if the side length is a?

7. If the side length of a regular hexagon is 5, what is the area? What if the side length is a?

8. The hexagons $ABCDEF$ and $ACGHJK$ are regular. Find the ratio of the area of the smaller hexagon to the area of the larger.

9. The area of rectangle $ABCD$ is 36. E, F and G are the midpoints of their respective sides $\overline{AD}$, $\overline{DC}$ and $\overline{CB}$. H is an arbitrary point on $\overline{AB}$. Find the sum of the areas of the shaded regions.



10. The difference in area between the larger square and the smaller square is 69 and the difference in their perimeters is 12. Find the dimensions of each square.