Berkeley Math Circle

September 15, 2015

1. There are n people in a room. Any group of $m \geq 3$ people in the room have a unique common friend. Can you determine n in terms of m?

2. Let $\mathcal{F} = \{E_1, \ldots, E_s\}$ be a family of subsets of a set X such that each set E_i has exactly r elements. If the intersection of any $r+1$ subsets in $\mathcal{F}$ is nonempty, show that the intersection of all subsets in $\mathcal{F}$ is nonempty.

3. Show that if we color the points of the plane in three distinct colors, we can always find two points of the same color at the distance 1 from each other.

4. Let G be a simple planar graph with all faces triangular. If the vertices if G are colored randomly in 3 colors, show that the number of faces whose vertices are colored in all the three colors is even.

5. At a chess tournament there are n participants. Knowing that any participant has to play against each of the other $n-1$ participants and that each player can play at most one game of chess a day, find the minimum number of days for the tournament.

6. Show that no matter how we choose three distinct integers, there are at least two of them, let's call them a and b, $a \neq b$, such that $a^3b - ab^3$ is a multiple of 10.

7. Given n distinct points in the plane, show that there are at most $n\sqrt{n}$ pairs of points that are at the distance 1 from each other.

8. (Magic square) We fill in a 3×3 square with non-negative integers such that the sum of the numbers on each row and each column equals n. Show that the number if such squares is:
$$\binom{n+2}{2}^2 - 3\binom{n+3}{4}.$$

9. The graph G consists of two odd cycles C_n and C_m with the vertices $A_1, \ldots, A_n$, respectively $B_1, \ldots, B_m$, as well as the edges A_iB_j for all $1 \leq i \leq n$ and $1 \leq j \leq m$.

   We color the $mn+m+n$ edges in either red or blue, such that no triangle has all three edges of the same color. Show that the $m+n$ edges of the two cycles C_n and C_m are either all colored red, or all colored blue.

10. We call a sequence of length $m+k$ **happy** if it has k letters a and m letters b and has the property that for any i, $1 \leq i \leq m+k$, among the first i letters we have at least as many letters a as letters b. Show that the number of happy sequences equals:

$$\binom{m+k}{k} - \binom{m+k}{k+1}$$

Note that for $m = k = n$ we get Catalan's number:

$$\frac{1}{n+1}\binom{2n}{n}.$$