BILLIARDS

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November 3, 2015

1 Introduction

The billiard theory in an active and exciting domain which is closely related to a variety of mathematical fields, like dynamical systems, geometry, group theory, complex analysis, and so on, and so on. Certainly this is a subject for books[1] or year long graduate course, rather than a two-hour presentation at a mathematical circle. My current presentation will consist of several loosely related to each other parts; they can be read and understood independently, except the construction discussed below in this Introduction, which will be used in many subsequent sections. I will demonstrate several exciting results of the theory. With a few exceptions, I will present almost (or entirely) no proofs: interested reader can fill some gaps using literature or Internet.

For a billiard table, we take an arbitrary polygonal domain in the plane. (We will also consider billiard tables bounded by arbitrary smooth curves, or "multistore" billiard tables not fully embedded into the plane: see Sections 6 and 7.) We assume that there are no pockets. Place a billiard ball at an arbitrary point and push it in some direction. The ball moves along a straight line until it reaches the wall (the boundary of the domain) where it reflects according to the usual rule *"the angle of incidence equals the angle of reflection"*. In the assumption of zero friction and infinite elasticity, the ball moves along a polygonal *billiard trajectory* which never ends (we always assume the ball to be measureless and the trajectory to never hit a vertex).

Notice that from the point of view of Dynamical Systems, the most natural question concerns *closed trajectories*. A closed trajectory is a trajectory which, after several reflection repeats itself. A closed trajectory looks like a closed polygon rather than an infinite polygonal line.

It is convenient to visualize billiard trajectories using the following *developments* (see Figure 1). We move along the trajectory, and when we reach the edge of the polygon, we reflect in this edge the whole polygon. The trajectory in extended to the reflected polygon as a straight line. When we reach the next edge, we again make a reflection of the (already reflected) polygon, and the trajectory is extended to the next polygon again as a straight

[1] Speaking of books, I can recommend the following: *S. Tabachnikov, "Geometry and Billiards"*, Student Mathematical Library, **30**, Amer. Math. Soc., 2005

line. Proceeding in this way, we convert our trajectory into a straight segment (or a half line, if it is infinite), and this segment is determined by and determines the trajectory.

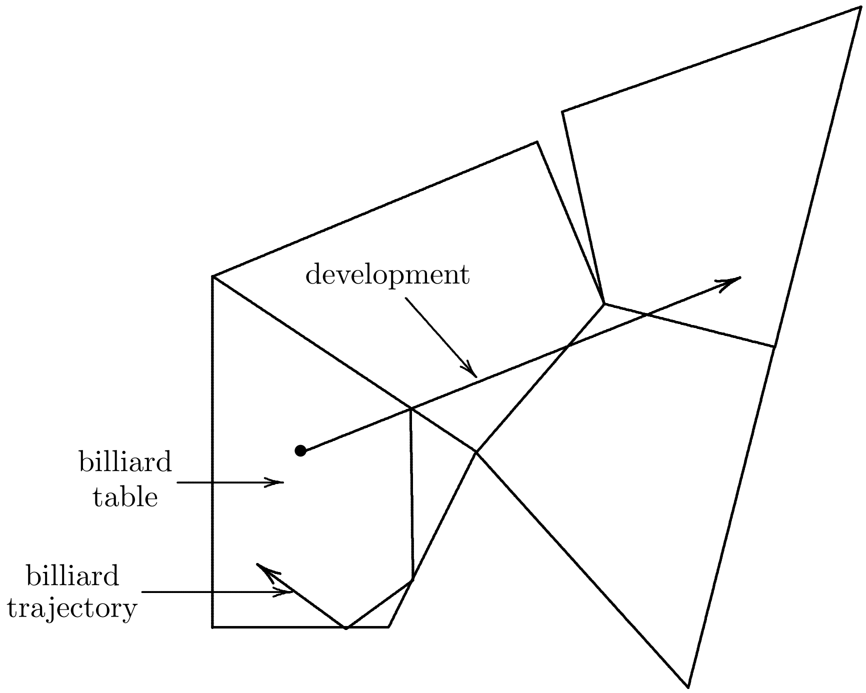

Figure 1: Development of a billiard trajectory

2 Billiard trajectories in a rectangular billiard

On a rectangular billiard table, mark two points, A and B. Let a billiard ball be at the point A. We want to push it in some direction in such a way that, maybe, after a certain (finite) number of reflections, it arrives at the point B. It is not to find all billiard trajectories starting at A and ending at B, and, in particular, to prove there are infinitely many such trajectories.

The construction is shown in Figure 2. We take our rectangle and reflect it in all its sides; then we do the same with the rectangles obtained, and so on. We obtain in the plane a rectangular lattice. We also apply all our reflections to the point B (and its images), so every cell of our lattice will contain an image of the point B. Now, if we join the point A (in the initial rectangle) with any of these points B, we will get infinitely many straight segment, and every one will be a development of some billiard trajectory (see Section 1) starting at A and ending at B, as shown in the bottom drawing in Figure 2.

This lattice construction also yields a classification of closed billiard trajectories in a rectangular billiard. Take an arbitrary (almost arbitrary: we do not want that the treajectory hits a vertex) starting point in the rectangle $ABCD$ (see Figure 3) and consider a billiard trajectory whose development has a "rational" direction, that is, a direction of the vector

$p \cdot \overrightarrow{AB} + q \cdot \overrightarrow{AD}$ where p and q are two integers, not both zeroes (we can assume them relatively prime).

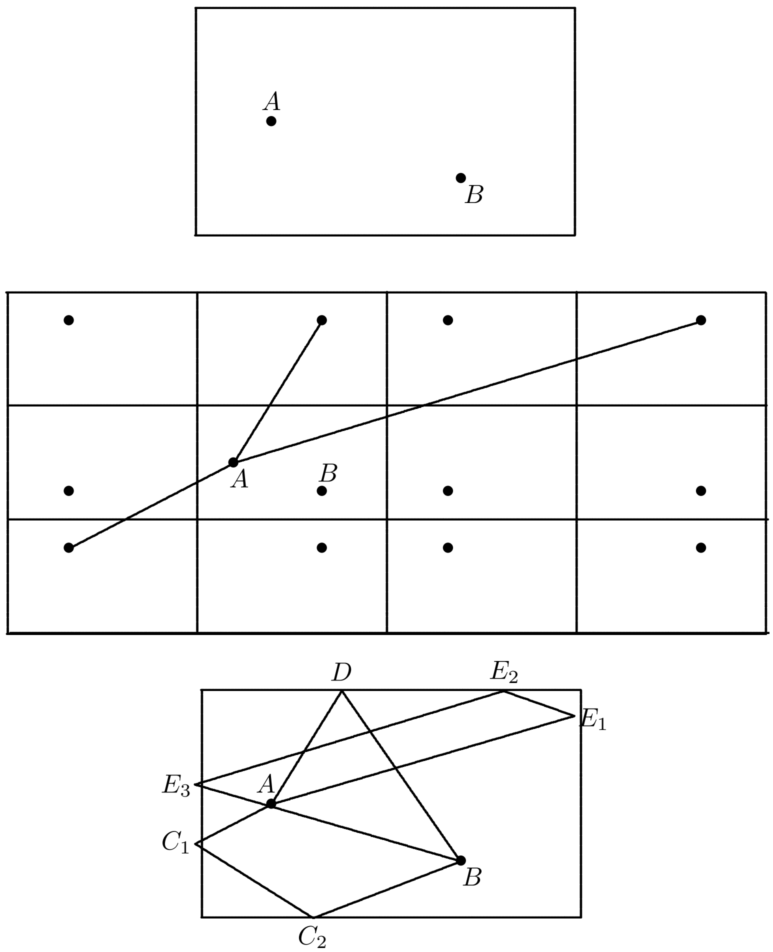

Figure 2: Billiard trajectories AB in the rectangular billiard

The trajectory in Figure 3 to the direction vector $3 \cdot \overrightarrow{AB} + 8 \cdot \overrightarrow{AD}$.

3 Triangular billiards

Next, we consider a billiard which has the shape of an arbitrary triangle. We will concentrate ourselves on one question: does this billiard has a closed trajectory. The answer shows how little our knowledge of polygonal billiards is: *it is not known*.

Still, there exists a classical construction (Giovanny Fagnano, 1755) of very short closed billiard trajectory for arbitrary *acute* triangles. It runs as follows.

Let ABC be an acute triangles, and let AD, BE, and CF be the three altitudes. Then DEF is a closed billiard trajectory (see Figure 4, left).

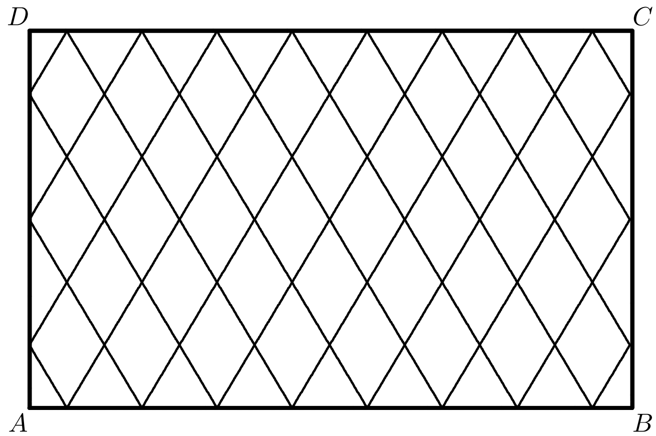

Figure 3: A closed billiard trajectory in the rectangular billiard

The proof of this result is shown in Figure 4, right. We need to show that the altitudes of $\triangle ABC$ are bisectors of the triangle formed by the bases D, E, F of the altitudes. Let α, β, and γ are the angles of $\triangle ABC$. Since the angles $\angle ADB$ and $\angle AEB$ are right, the points D and E belong to the semicircle with the diameter AB. From this, $\angle ADE = \angle ABE$ (two inscribed angles subtending the same arc) and, similarly, $\angle DEB = \angle DAB$. But $\angle ABE = 90° - \alpha$ and $\angle DAB = 90° - \beta$. This provides a computation of $\angle ADE$ and $\angle DEB$, and, in turn, proves the formulas for the six angles shown in Figure 4, left. This completes the proof of our statement.

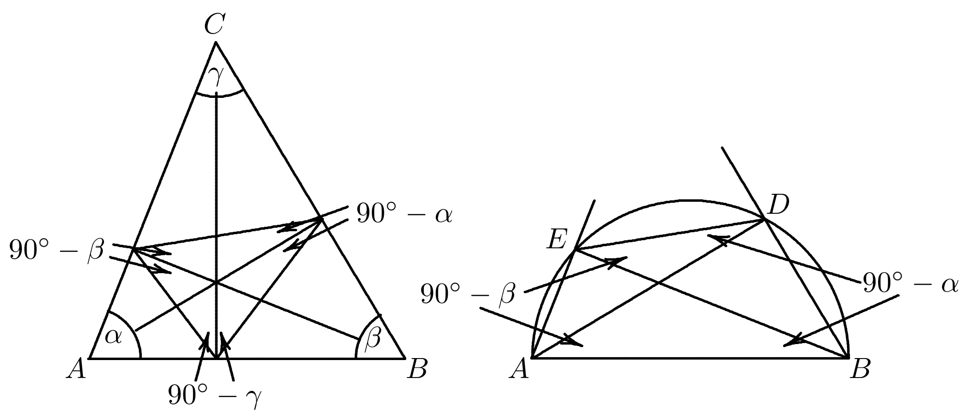

Figure 4: Fagnano closed billiard trajectory

There acute triangle possesses other closed billiard trajectories. A six-gonal one is shown in Figure 5. (We hope that the reader will understand the construction.)

The case of obtuse, and even right triangles is much more difficult. The best known result belongs to R. Schwartz who gave in 2008 a computer assisted but rigorous proof of the fact that if the obtuse angle of an obtuse triangle does not exceed 100°, then the triangle possesses a closed billiard trajectory. But this trajectory may be very long. R. Schwartz

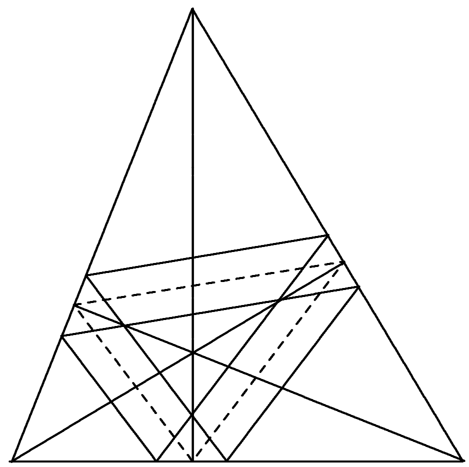

Figure 5: A "double Fagnano" closed billiard trajectory

proved also that for any L, in any proximity of any triangle with the angles $90°, 60°, 30°$, there exists a triangle whose shortest closed billiard trajectory has the length $> L$. Nobody has serious doubts that a closed billiard trajectory exists for any triangle; but no proof of that has been found, so far.

To finish this section on a more positive note, let us mention the following result. If all the angles of a polygon have, in degrees, rational measures, then this polygon has "many" closed billiard trajectories (for example, any choice of an initial point and a direction may be approximated by points and directions yielding closed trajectories). The proof of this has some similarity to the proof of this statement for rectangles (Section 2). We will observe this similarity in the simplest case: when the triangle is equilateral; this is the subject of the next section.

4 Closed billiard trajectories in an equilateral triangle

Similarly to a rectangular lattice, there exists the standard tiling of the plane by equilateral triangles. A billiard trajectory in the equilateral triangle is represented by a straight line development.

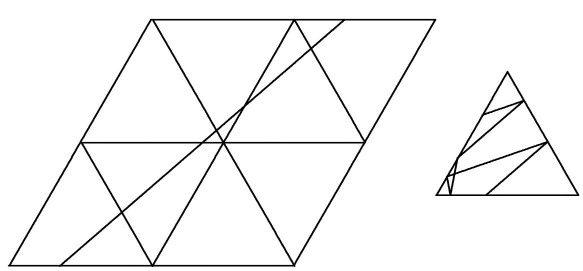

Figure 6: The development of a billiard trajectory in the equilateral triangle

To obtain a closed trajectory, we take two points in the triangularly tiled plane which occupy the same position in two parallel triangular tiles and join them by a straight segment. This sehment gives rise to a billiard trajectory which is either closed, or has ends on two different sides of the triangle. In this last case, the two endpoints are obtained from each other by a rotation of the triangle, This option is presented in Figure 6. To visualize this, we label the knots of the tiling by the letters A, B, and C, as shown in Figure 7, left. We see now that the trajectory begins at a point of the side AB and ends at the point of the side CA. To get a closed trajectory, we need to repeat the construction 3 times (Figure 7, right). The trajectory obtained is 18-gonal. It is clear that all closed billiard trajectories in the equilateral triangle can be obtained in this way.

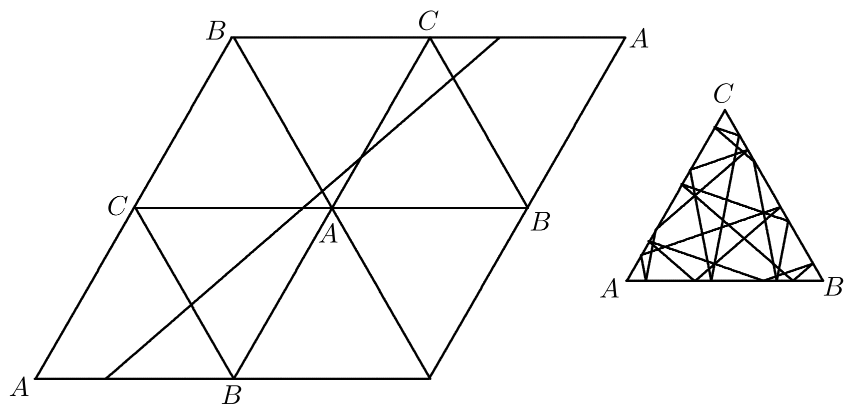

Figure 7: The same with knots labeled as A, B, C

Notice in conclusion that if we fold the triangle along the altitude, then the trajectory constructed above becomes a closed billiard trajectory in a $90° - 60° - 30°$ triangle (see Figure 8), and all closed trajectories in the $90° - 60° - 30°$ triangle are obtained in this way.

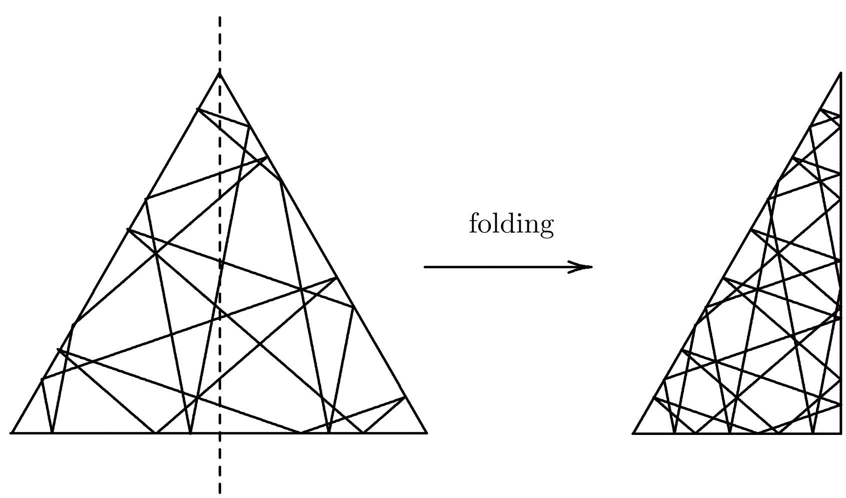

Figure 8: Closed billiard trajectory in the $90° - 60° - 30°$ triangle obtained by folding

5 Veech polygons: closed billiard trajectories in regular polygons

Consider two billiard trajectories in a polygonal billiard which have two different starting points, but the same initial direction; we call such trajectories *parallel*. Notice that parallel trajectories may look entirely different, even if they are close to each other at the beginning: as soon as a vertex of the polygon appears between the trajectories, they go different directions and are not close to each other any more.

A polygon is called a Veech polygon, if it has the following property: *there are some closed trajectories; if one of two parallel trajectories is closed, then another one is also closed.* In different words, the property of being closed of a billiard trajectory depends only on the initial direction. For example, it is not hard to deduce from the results of sections 2 and 4 that rectangles and equilateral triangles are Veech polygon. On the other hand, multiple examples show that in general, triangles are not Veech polygons.

A classification of Veech polygons is obtained in Veech's work of 1989. It is shown there, in particular, that *all regular polygons are Veech polygons.*

Our observations made in Section 2 show that rectangle have a property additional to the Veech property: not only two parallel trajectories are simultaneously closed, but if they are closed, then they have the same length. With rare exceptions (originated from the construction in Figure 5) parallel closed trajectories have the same length. Is this always true for Veech polygons? Unexpectedly, the answer to this question is negative.

Let us begin with the case of a regular pentagon. Consider a billiard trajectory starting at some point (it is convenient to assume that this point is taken on one of the edges) and forming, at the first moment, the angle θ with the side. The fact is that the trajectory arising will be closed if and only if $\tan\theta = (a + b\phi)\sin 36^\circ$ where a and b are rational numbers and $\phi = \dfrac{1+\sqrt{5}}{2}$ is the "golden ratio" (by the way, the proof is not elementary and rather hard). For every such θ, there arise a family of (parallel) closed trajectories, and the lengths of these closed trajectories have two values. The ratio of the two possible lengths is the same for all θ: it is ϕ. The simplest example is shown in Figure 9.

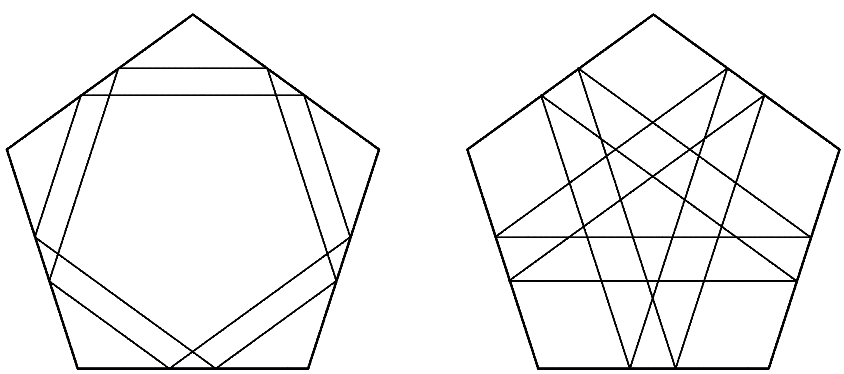

Figure 9: Parallel closed billiard trajectories of different lengths in a regular pentagon

If the length of the side of the regular pentagon in Figure 9 is 1, then the lengths of the two trajectories are 5ϕ and $5\phi^2 = 5 + 5\phi$. (Notice that to avoid two times shorter trajectories which arise when the reflection points on each side coincide, we need to assume that the trajectories do not hit not only the vertices, but also the midpoints of the sides.)

Let us formulate now the general fact. For any Veech polygon P, there exists a number N such that for every family of parallel closed billiard trajectories, there are precisely N different values for the lengths (again, sometimes we need to exclude a finite number of "degenerate" trajectories). Moreover, the ratios of these lengths do not depend on the direction. For example, for rectangles and equilateral triangles this number is one (same for $90° - 60° - 30°$ triangles and for regular hexagons); for regular pentagons this number is two (same for regular octagons). Another example: for regular heptagons, this number is three. Figures 10 and 11 present two triples of parallel closed billiard trajectories in a regular heptagon, which have different lengths.

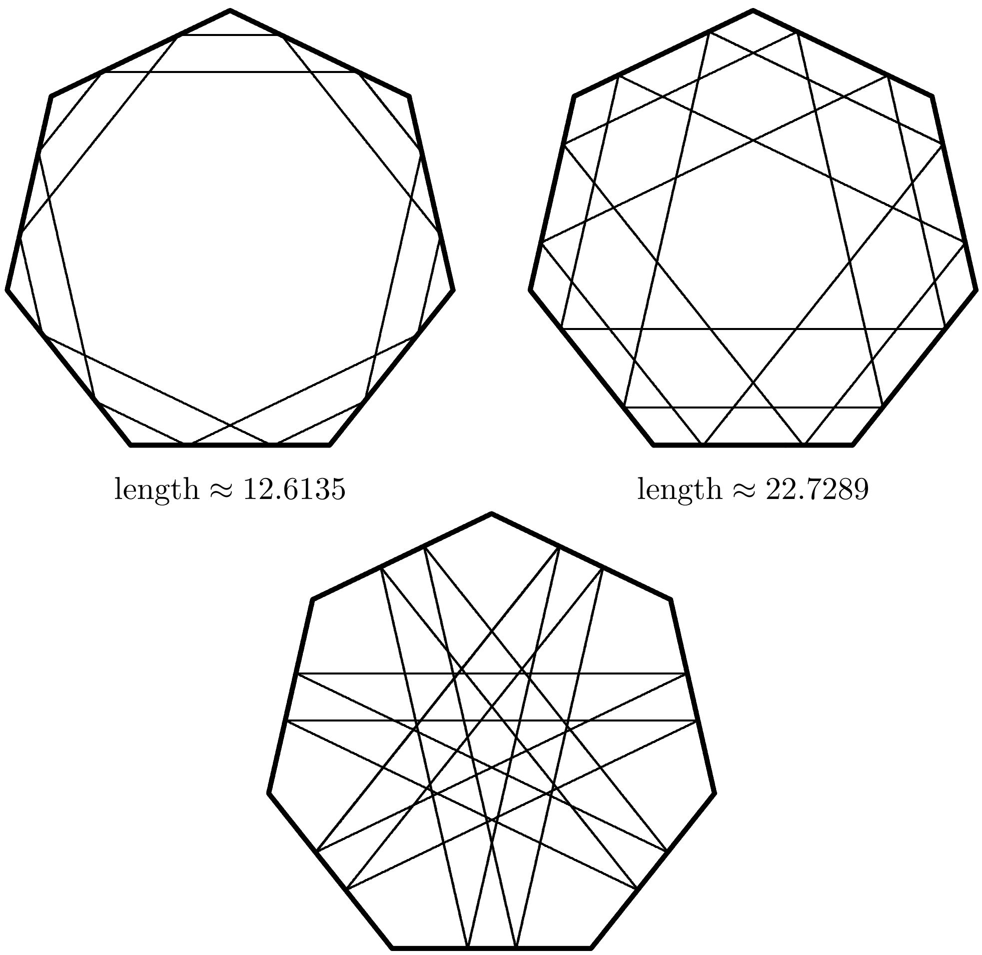

Figure 10: Parallel closed billiard trajectories in a regular heptagon

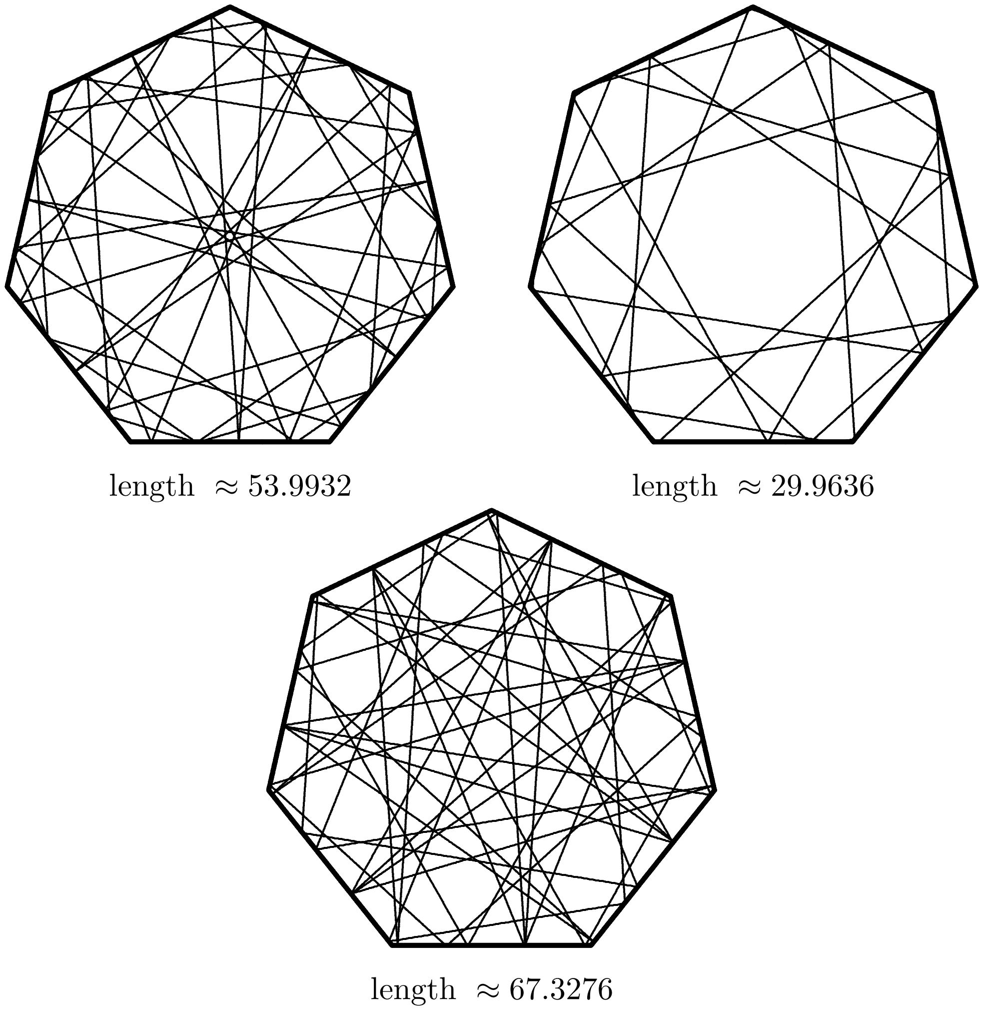

Figure 11: More parallel closed billiard trajectories in a regular heptagon

(In both cases, we assume that the length of the side of the regular heptagon is one.)

The ratios of the length do not depend on the direction. Namely, if the lengths of the thre trajectories are f, g, and h and $f < g < h$, then $h\colon g = 2\cos(\pi/7)$, and $g\colon f = 2\cos(2\pi/7)$.

6 Billiard trajectories in a right angle billiard

Consider now a billiard table, not necessarily convex, with all angles being right. Examples are shown in Figure 12.(By the way, the last figure shows a billiard table which is embedded in the plane with an overlapping.) Thus, every angle of our polygons is either 90° or 270°. This kind of billiards was considered in a recent work of Athreya, Eskin, and Zorich.

This kind of billiards was considered in a recent work of Athreya, Eskin, and Zorich. They considered the following problem. Take two vertices, say a and b. Let the angles at A

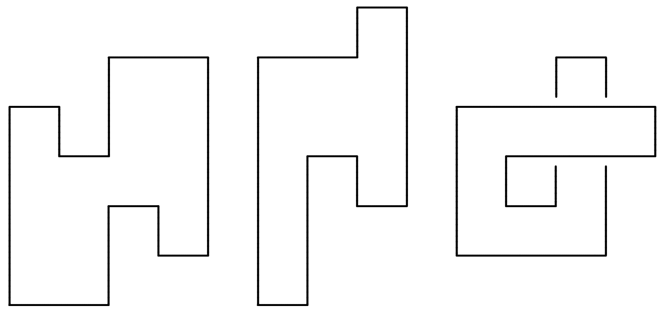

Figure 12: Right angle billiard tables

and B be α and β (thus each of α and β is either $90°$ or $270°$). For some (big!) number L, denote by $N(L)$ the number of billiard trajectories of length $\leq L$, which join a with b. How to estimate $N(L)$? The authors prove that the limit

$$\lim_{L\to\infty} \frac{N(L)}{L^2} \cdot \mathcal{A},$$

where $\mathcal{A}$ is the area of the billiard, is equal to:

$\frac{1}{2\pi}$, if $\alpha = \beta = 90°$;
$\frac{2}{\pi}$, if $\alpha = 90°$, $\beta = 270°$;
$\frac{16}{3\pi}$, if $\alpha = \beta = 270°$.

It looks absolutely unexpected that the result depends only on the area of the billiard, but not the shape of it.

7 Billiard trajectories in an elliptic billiard

Suppose now that our billiard is not polygonal, but is bounded by some smooth curve. There behavior of billiard trajectories in such billiards was studied a lot; still, not too much is known about it. A substantial information was obtained in the case, when the bounding curve is an ellipse. We will consider this case here.

I begin with a couple of pictures, and then will give some explanation. Each of Figures 13 and 14 presents one billiard trajectory: the left drawing presents a shorter piece of it, and the right drawing presents a longer piece. (Actually, both trajectories are infinite.)

We see that all the edges of the trajectory are tangent, in the first case, to a smaller ellipse, and in the second case, to a hyperbola. To say more about it, let me remind that ellipses, as well as hyperbolas, have *foci*. An ellipse is a locus of point with a given sum of distances from the foci; a hyperbola is a locus of points with a given difference of distances from the foci. Two ellipses, or an ellipse and a hyperbola, or two hyperbolas, may be *confocal* which means that they have the same pair of foci. Now the result:

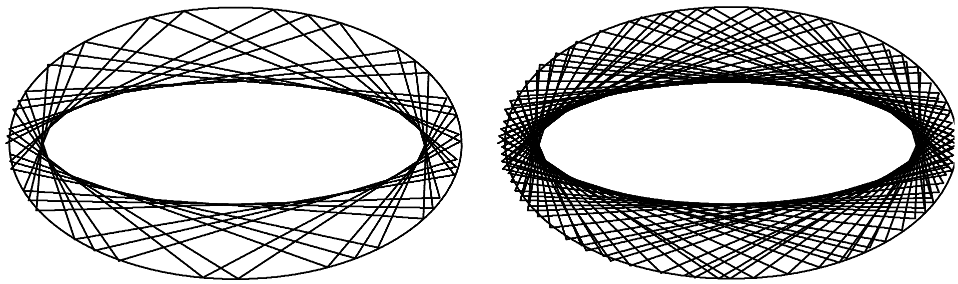

Figure 13: A billiard trajectory in an elliptic billiard

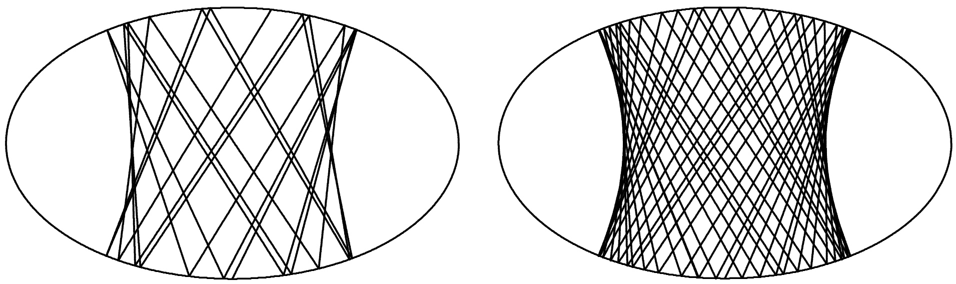

Figure 14: Another billiard trajectory in an elliptic billiard

All edges of a billiard trajectory in an elliptic billiard, which does not pass through the foci, *are tangent either to an ellipse or to a hyperbola confocal to the boundary ellipse of the billiard.*

How to learn which one? It is very easy. Assume that the starting point of the trajectory lies on the boundary of the table. Take the very first edge of the trajectory. If it passes between the foci, then the edges of the trajectory are tangent to a hyperbola, otherwise they are tangent to an ellipse.

And what happens, if the first edge passes through a focus? Then all the edges pass through foci: the first, the second, the first, the second, and so on. (I do not provide a picture, because it looks ugly.)