RATIONAL RENT-SPLITTING

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1. Introduction

Suppose when you get to college, you and a few friends move into an apartment together. You'd like to split the rent fairly, but there is a problem: not only are all the rooms different, but you each prefer different things. For example, you might be willing to pay much more to get the room with a private bathroom, while one of your friends cares more about being on the second story. Is there a way to split the rent so that everyone prefers a different room?

The answer is an emphatic yes. This result belongs to the field of fair division, which has been applied to divvying up unsatisfactory chores, to partitioning credit for joint accomplishments, and to slicing up a cake so that everyone is maximally happy with their piece. We are particularly interested in *envy-free* divisions, where things are partitioned in such a way that ever participant is maximally happy, and wouldn't want to trade with anyone else. The key will be a combinatorial lemma called *Sperner's Lemma*,

In this talk, we'll prove Sperner's lemma and elaborate on some of its applications. The vast majority of material presented here, including all of the figures except those in the last section, is borrowed from [1]. The result about splitting rent fairly has actually become somewhat well known, and was even the subject of a small New York Times piece; see

`http://www.nytimes.com/2014/04/29/science/to-divide-the-rent-start-with-a-triangle.html`

There is also an online app for splitting rent, which you can find at

`http://www.spliddit.org`

Sperner's Lemma can also be used to prove *Brouwer's fixed-point theorem*, a topological result of great importance in mathematics. One can use Brouwer's theorem to prove the following:

Theorem 1. *Place a map of Berkeley on a table. Then there is some spot on the map that is at the exact spatial coordinates as the spot it's supposed to represent.*

Another remarkable consequence of Sperner's Lemma is the following result:

Theorem 2 (Monsky's Theorem)**.** *It is not possible to dissect a square into an odd number of triangles of even area.*

We will prove both of these things.

2. Sperner's Lemma

Before beginning the main proof, we need some definitions. The *n-simplex* T_n is the set of points $(x_1, x_2, \dots, x_{n+1})$ in $\mathbb{R}^{n+1}$ given by the conditions $x_i \geq 0$ for all i, and $\sum_{i=1}^{n+1} x_i = 1$.

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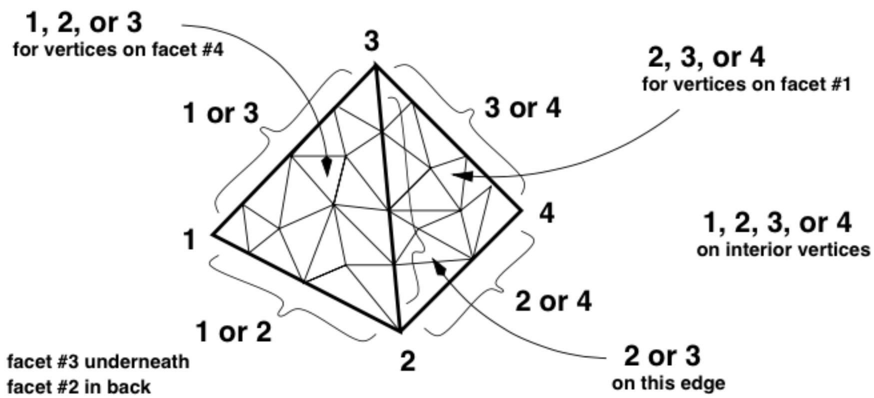

FIGURE 1. A Sperner tetrahedron.

(More generally, we can also refer to the convex hull of any $n+1$ points in $\mathbb{R}^m$, $m > n$, as an n-simplex).

A *k-face* of T_n is a subset spanned by any k of the vertices, and an n-face is called a *facet*. A triangulation of T_n is a subdivision into subsimplicies, such that if two subsimplicies intersect, their intersection is a shared facet. A *vertex* of the triangulation is just a vertex of one of the subsimplicies.

Lemma 1 (Sperner's Lemma)**.** *Consider a triangulation of an n-simplex in* $\mathbb{R}^{n+1}$ *and label each vertex of this triangulation with one of* $1, 2, \ldots, n+1$*, subject to the condition that a vertex lying on a face with vertices* $a_1, a_2, \ldots, a_n$ *is itself labeled by one of* $a_1, a_2, \ldots, a_n$*. (Internal subvertices have no restrictions on their labels). Then the number of subsimplices whose vertices are labeled* $1, 2, \ldots, n+1$ *is odd. In particular, it is nonzero.*

Proof. We proceed by induction on the dimension n. The case $n = 1$ is left as an exercise to the reader.

Now consider T_n with a triangulation and a Sperner labelling with labels $1, 2, \ldots, n+1$. This induces a Sperner labelling on the face F spanned by $1, 2, \ldots, n$, so we know there are an odd number of $n-1$-simplicies on this face with labels $1, 2, \ldots, n$.

Let us call a face of a subsimplex "good" if it has the labels $1, 2, \ldots, n$. We have shown that there are an odd number of boundary subsimplicies with a good face. The key observation is that a subsimplex can have at most 2 good faces, and if it has only one good face, then it's a fully labelled subsimplex.

So start at a boundary subsimplex S_1 with a good face F_1. If it has no other good face then it's completely labelled and we're done, otherwise it must have another good face F_2. Now this face is shared with another subsimplex S_2. If S_2 has no other good faces then it's completely labelled and we're done, otherwise it has another good face F_3 that is shared by another subsimplex S_3. Continuing in this way, we trace a path through the subsimplices that must either terminate at a fully labelled subsimplex, or that must lead us back to another boundary subsimplex. So in this way, we can "pair up" boundary subsimplices with good faces. But by the inductive hypothesis there are an odd number of such subsimplicies, so there must be an odd number of them whose paths lead to fully labeled subsimplicies. We can also repeat this process to pair up fully-labelled subsimplicies that are not reachable by paths from the boundary. So the number of fully labelled subsimplicies is odd. □

The above proof was constructive, in that it gives an algorithm for actually locating a fully-labelled subsimplex. It also suggests the following nonconstructive proof. Let $f(S)$ denote the number of good faces of subsimplex S, and consider the sum $\sum_i f(S_i)$ over all subsimplices $S_1, S_2, \ldots, S_M$. Each interior good face is counted twice in this sum, and each boundary good face is counted once. Since there are an odd number of good faces on the boundary this sum is odd, which means an odd number of terms in the sum are odd, which means there are an odd number of fully-labelled subsimplicies.

3. Cake cutting

Suppose we have a cake that we would like to divide among n people. We can't just cut the cake into n pieces of equal area, because everyone might have a different piece of the cake they like the most. Maybe you really want a slice with lots of chocolate icing, while your friend cares most about getting a frosting flower. Is there a way to cut it up so that everyone is happy?

A solution for $n = 2$ is the famous "I cut, you choose" method. For larger n, one approach is given by the *moving knife algorithm*. This proceeds as follows. Take a knife and hold it above and to the side of the cake. Now slowly sweep it above the cake. Whenever someone feels that the knife would cut off a piece of value $\frac{1}{n}$ they shout "stop!", and you cut the cake at that point and give them that piece. Then that person leaves the table, and you repeat the process with the remaining players and the remaining cake.

We say that a player thinks his piece of cake is "fair" if he assigns it a value of at least $\frac{1}{n}$ (that is, he thinks he's getting his fair share of the total cake),

Exercise 1. *Show that this results in everyone getting a piece of cake that they think is fair.*

The problem with this approach is that while everyone ends up with a cake they value at $\frac{1}{n}$, someone might still prefer another player's slice to their own. This is because when they yell "stop" and get their slice, they don't know how the rest of the division will turn out.

What we actually want is an envy-free approach, where everyone is happiest with their piece. To put this on mathematical footing, consider the set of cuts where each cut is parallel to the left side of the cake. Assume the cake is of unit length, and let $x_1, x_2, \ldots, x_n$ be the lengths, in order, of the cut pieces. Then $(x_1, \ldots, x_n) \in T_{n-1}$ is a point in the $n-1$-simplex. We call such a division a *cut set*, and we call such a set envy-free if we can assign a different piece to each player such that everybody at least prefers their piece to all others. (Note that some players may prefer multiple pieces equally).

We now assume

- (Boundary) Players always prefer a positive amount of cake to no cake (they're hungry!), and
- (Continuity) If a player prefers a series of pieces in a limiting sequence of cut-sets, then she prefers the piece in the limit.

Under these mild and plausible conditions, we have the following:

Theorem 1. *There exists an envy-free cake division.*

Proof. Given $\epsilon > 0$, construct a triangulation of T_n such that the vertices of every subsimplex are all within ϵ of each other, and such that each vertex can be labeled with a name of one of the players such that every subsimplex has vertices labeled with the names of all players. (This can be done by taking a sufficiently fine mesh, and then passing to the barycentric subdivision of that mesh). Now for each vertex, ask the corresponding person what piece of cake she would prefer at that cut set, and label the vertex with that number. By condition (1) this is a Sperner labelling, so there must exist a fully labeled subsimplex. By compactness, in the limit as $\epsilon \to 0$ there must be a sequence of such subsimplices that converges to a single point, and by property (2), this point is an envy-free subdivision. □

4. Rental Harmony

We now return to the problem posed at the beginning and show how Sperner's Lemma can be used to partition rent so that everyone is happy.

As with cake-cutting, we require that a couple condition be satisfied:

- (Boundary) If some rooms are free, everyone will prefer one of those rooms to any other room.
- (Continuity) If a player prefers the same room in a limiting sequence of rental divisions, then he prefers that same room in the limit.

We then have the following:

Theorem 2. *There exists a partitioning of rent such that each player prefers a different room.*

Proof. Given a partition of rent, let $x_1, x_2, \ldots, x_n$ be the proportion of rent for each of the n rooms. Then $x_i \geq 0$ and $\sum_{i=1}^{n} x_i = 1$, so the set of such divisions forms a simplex. As in the proof of Theorem 1, for each $\epsilon > 0$, construct a triangulation of the simplex with the property that the vertices of each subsimplex are of distance less than ϵ, and such that we may assign a player to each vertex so that every subsimplex has vertices associated to all the players. Now for each vertex, ask the owner of that vertex what room she would prefer given that rent partition, and label the vertex with that room number.

Here is where the proof diverges from the original. This is *not* a Sperner labelling! It is actually a dual Sperner labelling: if the vertices of the original simplex are $a_1, a_2, \ldots, a_{n+1}$, then by the boundary condition, each vertex in a space spanned by $a_{i_1}, a_{i_2}, \ldots, a_{i_k}$ must be labelled by a_j for $j \neq i_1, \ldots, i_k$. Fortunately, we have the following:

Exercise 2. *Show that a dual Sperner labelling has an odd number of fully labelled subsimplicies.*

So by taking finer and finer meshes, we see there must exists a series of fully-labeled subsimplicies that tend to a single point. By continuity this point is a fair rental division. □

5. Brouwer's Fixed Point Theorem

We will use Sperner's Lemma to prove the following (famous) result.

Theorem 3. *Let B_n be the n-dimensional unit ball in $\mathbb{R}^n$. Then for every continuous function $f : B_n \to B_n$ there exists $x \in B_n$ such that $f(x) = x$.*

The point x is known as a *fixed point.*

Note that the case $n = 1$ can be solved by an application of the intermediate value theorem, but the proof gets much trickier for larger n. The usual argument involves topology and some higher mathematics. But not today!

Proof. Since B_n is homeomorphic to T_n, it suffices to prove the statement for maps $f : T_n \to T_n$.

To each vertex v in our triangulation, if $v' = f(v)$, assign to v the smallest number i such that $v_i > v'_i$. (Since the sum of the coordinates of both v and $f(v)$ are 1, if $v'_i \leq v_i$ for all i then $v = v'$ and v is a fixed point). Let $a_i \in T_n$ be the vertex of the simplex whose i^{th} coordinate is 1 and whose other coordinates are 0. Then a_i has color i. More generally, if v is on a face spanned by $a_{i_1}, \ldots, a_{i_k}$, then v must be colored one of $i_1, \ldots, i_k$.

So this is a Sperner coloring, and there must exist some fully-labelled subsimplex. By taking a series of successively finer triangulations we get a sequence of fully-labelled subsimplicies which converge to a point x, and this is our fixed point. ☐

We now give the proof of Theorem 1. Let $f : M \to M$ be the function that assigns to each GPS coordinate x on Berkeley's campus the GPS coordinate of the physical point on the map corresponding to x. Then by the above theorem f has a fixed point, where the position on the map is exactly in the same spot as the position it represents.

6. Monsky's Theorem

6.1. **Introduction.** In this section, we turn to the problem of dissecting a square into n triangles of equal area. Such a dissection is easy when $n = 2$, and a bit of tinkering shows one can also do it for $n = 4$ and $n = 6$. This might lead you to suspect that one can always do this for even n, and in fact this is so.

Exercise 3. *Show that for all even numbers n, a square can be dissected into n triangles each of area $\frac{1}{n}$.*

Such a dissection is obviously impossible when $n = 1$, and if you try $n = 3$ you will also run into difficulties. In this section, we will sketch a proof of the following theorem:

Theorem 4 (Monsky's Theorem)**.** *It is not possible to dissect a square into an odd number of triangles of even area.*

The rough outline of the proof is as follows. Let T be a dissection of the square into n equal-area triangles for n odd. We will introduce a certain function $\mathbb{R}^2 \to \{\text{red}, \text{blue}, \text{green}\}$ that assigns a color to every point in the plane, and we use this to color the vertices of T. This coloring has the special property that every rainbow triangle – that is, every triangle with vertices of all three colors – cannot have area $\frac{1}{n}$ for n an odd integer. But by a Sperner-style argument, T must have at least one rainbow triangle, contradiction.

It is worth mentioning that while the idea behind Sperner's Lemma is instrumental in this proof, it is only one of a series of key insights that are needed. For that reason, this section is a significant departure from the rest of this presentation. However, the result is so spectacular that I thought it was worth including. This entire section is based on the material in [3].

6.2. **The p-adics.** To define the coloring function, we need the notion of a *norm*

Recall the absolute value function $|\cdot| : \mathbb{R} \to \mathbb{R}_{\geq 0}$, which has the following properties:

(i) $|x| = 0$ if and only if $x = 0$,
(ii) $|xy| = |x||y|$, and
(iii) $|x + y| \leq |x| + |y|$.

Now consider the following definition.

Definition 1. *A function $v : K \to \mathbb{R}_{\geq 0}$ on a field K is called a* non-Archimedean norm *if the following hold:*

(i) $|x| = 0$ if and only if $x = 0$,
(ii) $|xy| = |x||y|$, and
(iii) $|x + y| \leq \max\{|x|, |y|\}$,

for all $x, y \in K$.

Note that the usual absolute value function satisfies (i) and (ii); it is condition (iii) (known as the *ultrametric inequality*) that makes a norm non-Archimedean.

Exercise 4. *Show $|1| = 1$ and $|x^{-1}| = |x|^{-1}$.*

A natural question is: what is an example of such a norm over $K = \mathbb{Q}$, the field of rationals? Let p be a prime, and define the function $|\cdot|_p$ as follows: For any rational $x \in \mathbb{Q}$, we can write $x = p^k \frac{a}{b}$ for some integers k, a, b, where $\frac{a}{b}$ is in lowest terms. Then let $|x|_p = p^{-k}$.

Exercise 5. *Show that $|\cdot|_p$ is a norm for all primes p.*

This norm is known as the *p-adic* norm. A famous theorem of Ostrowski says that in fact, *every* nontrivial real non-Archimedean norm of $\mathbb{Q}$ is either a p-adic norm or a power of one.

The proof of Theorem 4 requires a norm $|\cdot|$ on $\mathbb{R}$ with the property that $\left|\frac{1}{2}\right| > 1$. The 2-adic $|\cdot|_2$ has this property, but it is only defined over $\mathbb{Q}$. However, a famous theorem of Chevalley says that for every prime p, there exists an *extension* $|\cdot|$ on $\mathbb{R}$ with the property that $|x| = |x|_p$ when $x \in \mathbb{Q}$. We do not give the proof here, though we do require the existence of an extension of $|\cdot|_2$.

So from now on, "norm" refers to this extension.

Exercise 6. *Let $|\cdot|$ be a norm on $\mathbb{R}$ with the property that $\left|\frac{1}{2}\right| > 1$. Show that $\left|\frac{1}{n}\right| = 1$ for all odd integers n.*

6.3. **An unusual coloring.** Color the points (x, y) in the unit square by looking at the entries of the triple $(x, y, 1)$, and seeing which entry has the largest norm. In particular, we assign a color to (x, y) as follows:

$$(x, y) \text{ is colored} \begin{cases} \text{blue} & \text{if } |x| \geq |y|, |x| \geq |1|, \\ \text{green} & \text{if } |x| < |y|, |y| > |1|, \\ \text{red} & \text{if } |x| < |1|, |y| < |1|. \end{cases}$$

Exercise 7. *Show that for any blue point $p_b = (x_b, y_b)$, green point $p_g = (x_g, y_g)$ and red point $p_r = (x_r, y_r)$, the norm of the determinant*

$$\left| \det \begin{pmatrix} x_b & y_b & 1 \\ x_g & y_g & 1 \\ x_r & y_r & 1 \end{pmatrix} \right|$$

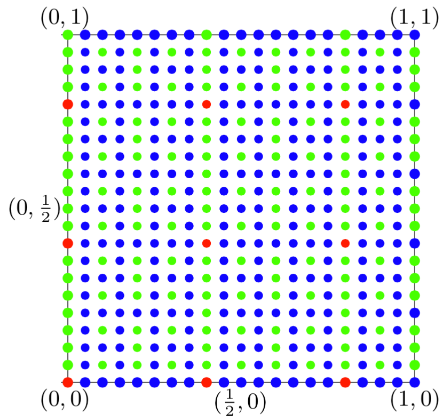

FIGURE 2. Colors of points of the form $(\frac{a}{20}, \frac{b}{20})$ for integers $0 \leq a, b \leq 20$.

is at least 1*.*

That determinant D is twice the area A of the rainbow triangle with vertices p_b, p_g, p_r, so if $A = \frac{1}{n}$, then $D = \pm\frac{2}{n}$ and

$$\left|\pm\frac{2}{n}\right| = \left|\frac{1}{2}\right|^{-1}\left|\frac{1}{n}\right| < 1,$$

a contradiction. Note also that p_b, p_g, p_r cannot be collinear, or else $A = 0$ which would give $D = 0$. We have thus shown that:

Corollary 1. *Any line of the plane receives at most two different colors. The area of a rainbow triangle cannot be* 0*, and it cannot be* $\frac{1}{n}$ *for* n *odd.*

Now we are close. Consider a triangulation of the unit square, and color the vertices of the triangulation as above. The bottom of the square will only have red and blue points, the left side will have green and red points, and the top and right sides will only have blue and green points. Give a Sperner-style counting argument to prove the following.

Exercise 8. *Show there exists an odd number of rainbow triangles. In particular, there exists at least one.*

So the triangulation must contain a nondegenerate rainbow triangle. But by Corollary 1 triangle cannot have area $\frac{1}{n}$ for n odd, contradiction.

REFERENCES

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[3] Aigner, Martin and Ziegler, Gunter M. “Proofs from THE BOOK.” Springer 2009.