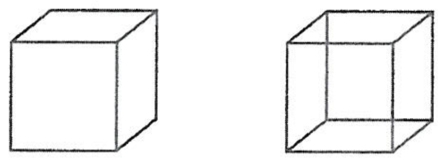
November 17, 2015

BMC Worksheet. 3D Geometry: Meet the Cube.

Today we are going to examine the cube from all different points of view, and work on problems about the cube.

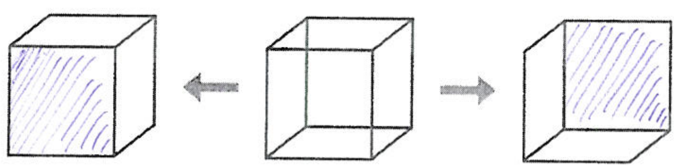
First, let's look at the following two cubes. What do they tell us? How are they different?



The second drawing shows what's called a *wireframe model* of the cube: what's left of the cube if you remove the faces and the interior, and leave only the edges. (Alternatively, you can think of the faces as transparent.)

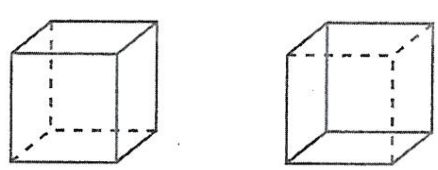
The solid cube on the left is very realistic, and it is perfect for a drawing lesson. However, it has one disadvantage: it is impossible to see the hidden edges and back faces of this cube. Therefore, in our math classes we generally stick to the wireframe model.

But the wireframe drawing in the figure above is not perfect either. Can you tell which is the front face? Yes and no, because there are two possible answers:



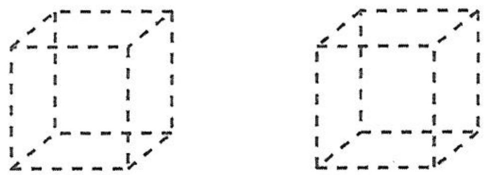
Depending on where you focus your eyes, you can see two different cubes: one with its front face on the lower left, and one with its front face on the upper right.

The best solution is to draw the invisible edges with dashed lines instead of solid lines. So instead of the ambiguous wireframe drawing in the middle of the preceding illustration, we use one of the following drawings:

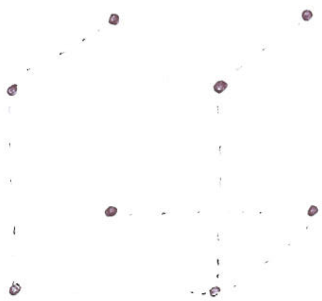


"Meet the Cube" Exercises

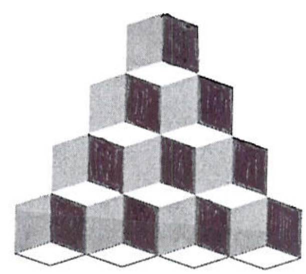
- **Exercise 1.** Complete each of the two identical cube templates below into a different view of a cube:



Try again a few times:



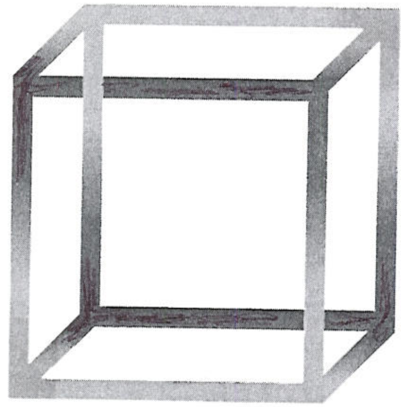
- Exercise 2. In the picture on the right, you can see either 6 or 10 cubes. You will see 6 cubes if you view the white faces as the top sides of the cubes. You will see 10 cubes if you view the white faces as the bottom sides of the cubes.



Did you see 6 cubes? _____

Did you see 10 cubes? _____

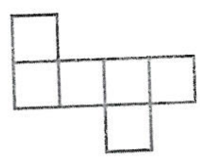
• Exercise 3: What do you see on this picture?



This picture displays a 2D projection of an impossible 3D cube. Why is this cube impossible? If you cover the right half of the image you see the left half of a cube; if you cover the other half, you see the right half of a cube. Both halves of a cube are perfectly correct, but they can't be reconciled. (Try also the top and bottom halves!)

Cube Nets

Our next discussion topic is about two-dimensional figures that can be folded into three-dimensional objects.

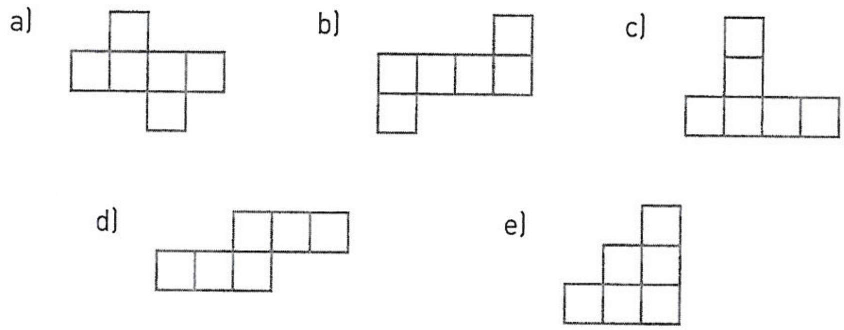
• Exercise 4: Can this shape be folded into  a cube in such a way that each square becomes a face of the cube?

• This one? 

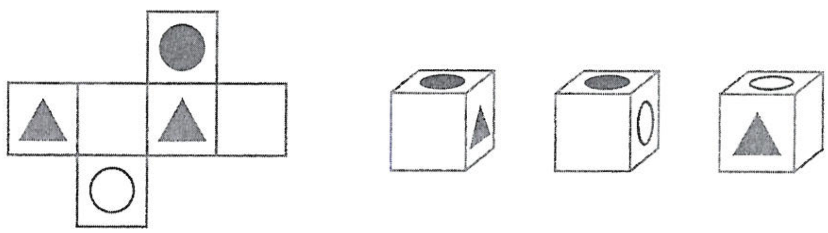
DEFINITION 1. A *net* is a two-dimensional shape that can be folded into a three-dimensional body.

DEFINITION 2. A *cube net* is a two-dimensional shape that can be folded into a cube.

- **Exercise 5.** Which of the following 2D shapes are cube nets (that is, can be folded into cubes)?



- Exercise 6. Which of the cubes below could have been folded from the cube net sketched on the left?



Cross Sections

- What exactly is a cross section?

- The cross section is the new flat 2D surface that was created by the cut.

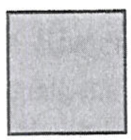
- So, what kind of a cross section can we get when we cut an apple? If we are careful enough not to cut too close to the apple's core, the newly created cross section will always be in the shape of a circle.

– To get a cross section of an object, you are allowed to make just *one straight cut* that goes *all the way through*: do not stop halfway through, do not change the tilt of the knife in the middle of the cut, and do not carve fancy shapes by making two or more cuts.

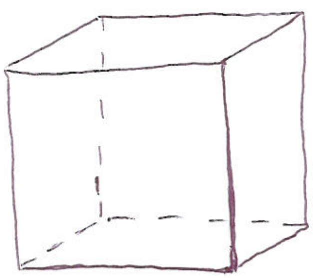
– The cross section is the flat 2D face that was opened up by the cut, not the 3D chunk that was cut off.

- Exercise 7. What kinds of cross sections can we create from a sphere with one straight cut of a knife?

- Exercise 8. Suppose you make a single straight knife cut through a solid cube. The resulting cross section will clearly be some sort of a polygon. Let's pose several questions about this cross section.
  - Can you cut this cube in such a way as to get a square cross section?



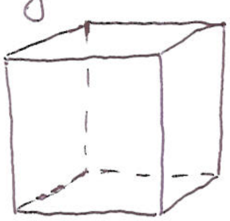
← this would be a cross section.



← Draw a cross section

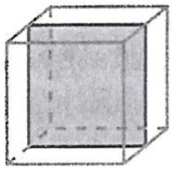
  - How about a rectangular cross section that is not a square?

Try drawing a cross-section in this cube!

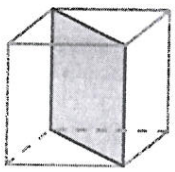


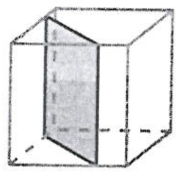
Is it possible to get any other cross sections, which are rectangular?

- Here is a square cross section:



- Here are possible rectangular cross sections:





– How about a triangular cross section?
– A cross section in the shape of trapezoid?
– A cross section in the shape of a polygon with more than 4 vertices?

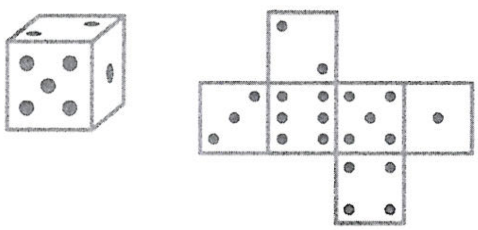
Draw all the cross sections that you come up with.

Practice (and Homework) Problems

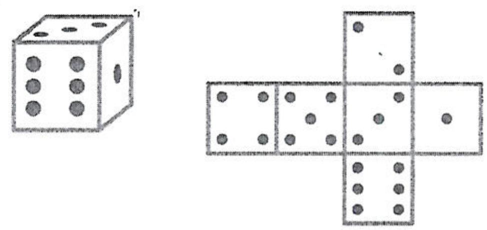
Problem 1. In the figure you can see two dice and their 2D nets. Which number is on the:

- Bottom face of each die?
- Rear (hidden) face of each die?
- Left (hidden) face of each die?

a)

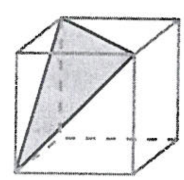


b)



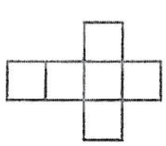
Problem 2. A cube has been placed on the table. A lamp is hanging directly above this cube. The cube is balanced on one vertex, with the opposite vertex pointing up. What does the shadow of this cube look like?

Problem 3. The triangle in the figure has its vertices at the corners of the cube. Find the angles of this triangle.

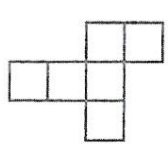


Problem 4. Which of these shapes are cube nets (that is, can be folded into cubes)?

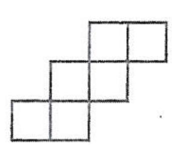
a)



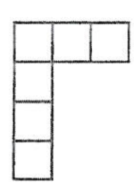
b)



c)



d)



Problem 5. (a) Is it possible to cut a square into 10 smaller squares (not necessarily of equal size)? If so, show how. If not, explain why. (Squares that are subdivided into smaller ones don't count.)

(b) Same question for 11 squares.

Problem 6. Snow White decided to make a quilt. She took a big square piece of fabric and cut it into 4 equal squares. She left them on the table and went to the kitchen. The first dwarf, who happened to pass by the table, picked one square at random. He cut it into four equal squares and placed them back on the table. All the other dwarfs (the second, the third, the fourth, the fifth, the sixth and the seventh) did the same thing: each picked a random piece of fabric and cut it into four smaller squares. How many pieces of fabric were there on the table when Snow White came back to the room?

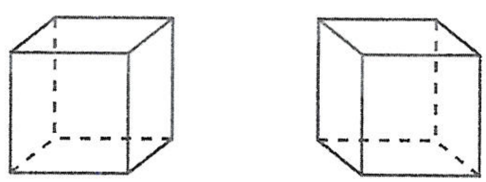
Problem 7. A lame but merry cricket jumps along a straight line. On every jump, it leaps 3 inches to the right or 5 inches to the left.

(a) Is there a way for the cricket to end up 1 inch to the right of its starting point? How about 1 inch to the left? If the answer is yes, show the sequence of jumps. If it is no, explain why.

(b) The cricket made 20 jumps. Is it possible for it to end up 1 inch to the right of the starting point?

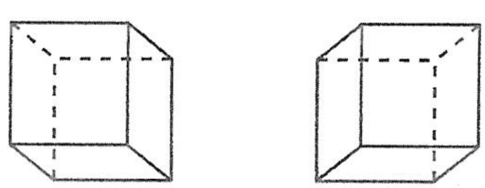
(c) The cricket made 23 jumps. Is it possible for it to end up 10 inches to the left of the starting point?

Problem 8. (a) Grab a cube; try to position your hand and orient the cube in such a way as to be able to view the cube as it is displayed on the left picture below:

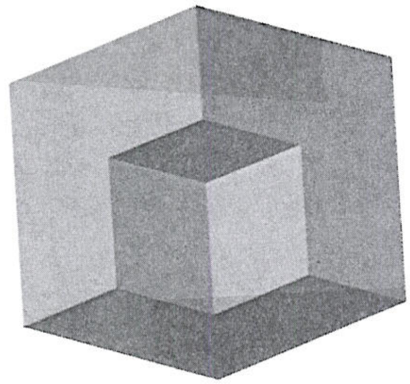


(b) Repeat the same exercise for the right picture.

Problem 9. Repeat the same exercise for the drawings shown here:



Problem 10. What do you see here?



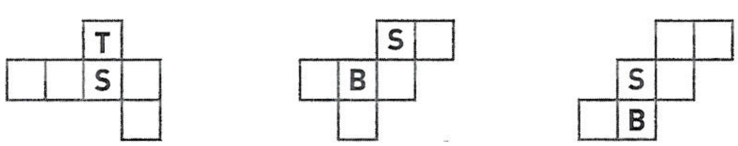
One way to interpret the picture is to view it as a cube that has been placed on the floor in the corner of a room.

Another interpretation is to treat it as a cube that is hanging above you, with a smaller cube carved out from the bottom-front corner. (The dark V-shaped polygon serves as the bottom surface of this altered cube). In addition, there are other ways to interpret this picture.

Problem 11.

Marian made several 3D paper models of the cube from cutouts. On each assembled cube, she labeled the faces with letters: T for the top, B for the bottom, and S for all the side faces. Next, Marian unfolded the cutouts to store them.

Later Marian's younger sister, Bella, erased some of the letters. Here is what they looked like at that point:



Please restore the missing letters.

November 17, 2015

Selected Answers to BMC Worksheet: Odd and Even Numbers (from Nov. 10)

13. In the morning, there were 5 spaceships at the spaceport of the planet Pandora. During the day, several more spaceships landed, and a few spaceships departed from the port. The ships always arrived in groups of 2 or 4, and departed in groups of 2. In the evening, the dispatcher counted 60 spaceships at the spaceport. Prove that the dispatcher miscalculated the number of spaceships. In the morning, the number of spaceships was odd. Every time a group of spaceships arrives, this number goes up by an even number. Every time a group of spaceships departs, the number goes down by an even number. Since the sum of an odd and an even number is odd, and the difference between an odd and an even number is odd as well, the number of spaceships remains odd. The dispatcher's result is even. Therefore, he made a mistake.

14. There are three towns in a county: A, B, and C. The residents of town A never lie, those from town B never tell the truth, and those form town C alternate true and false statements.

    One day, the county fire station received a call: "There is a fire in our town!" When the firefighter asked where the fire was, he got the reply, "In town B." Which town should the fire fighters go to? (Assume the call was from a resident of the town where the fire was.) The call could not have been placed from town A. If it were, then the caller would report that the fire was in town A. Likewise, the call could not have been placed from town B, since then Town B would not have been reported as the location of fire. Thus, the call must have been placed from C; out of the two statements, the first one (about the fire) was true, and the second one (about the location) was false.

15. Peter says, "The day before yesterday I was 10, but next year I will turn 13." How can this be possible if we know that Peter is not lying? Peter has his birthday on Dec 31.

16. Johnny the Junior Hacker reprogrammed the elevator in the 100-story Boogle Corporation building: only two buttons are currently working. The first button sends the elevator 8 floors up, and the second one 6 floors down. (The elevator will not move if it is asked to go above the 100^{th} floor or below the 1^{st} floor.)
    a. The company's CEO is currently drinking coffee on the first floor. (There is no lobby floor in the building.) Can he take the elevator to the 95^{th} floor? If so, show how. If not, explain why. An incorrect solution of this problem sends the elevator beyond the 100^{th} floor. For example, repeating 8 floors up, 6 floors down combination until the elevator stops on the 95 floor. In order to get to the 95^{th} floor by pressing the 6 floors down button you would have to be on the 101^{st} floor. Pressing the 8 floors up button 12 times puts the elevator on the 97^{th} floor. Pressing the 6 floors down button 3 times puts the elevator on the 79^{th} floor. Then pressing the 8 floors up button twice puts the elevator on the 95^{th} floor.

    b. Can he take the elevator to the 96^{th} floor? If so, show how. If not, explain why. It is not possible to get to the 96^{th} floor on the elevator. The elevator always stops on an odd numbered floor since it starts on an odd floor (1^{st} floor) and always goes up or down an even number.

17. Julia is walking home from school. She left the school 5 minutes earlier than her next-door neighbor, Josh. However, Josh is in a hurry because he wants to give Julia the cell phone that she left at school. Josh is walking 1.5 times faster than Julia. How soon will Julia get her cell phone back? After 10 min

18. Bobby the Beaver is cutting several long logs into smaller pieces using a chainsaw (all cuts are across a log). Bobby made 30 cuts and ended up with 36 pieces of wood. How many long logs did he start with? If the beaver had just one log, the number of pieces would always be 1 more than the number of cuts. If he had 2 logs, the number of pieces would always be 2 more than the number of cuts. If he had 3 logs, the number of pieces would always be 3 more than the number of cuts. And so on. Since the number of pieces (36) is 6 more than the number of cuts(30), the beaver must have started out with 6 logs.

19. Schmerlin the Magician inherited a beautiful old book of spells. The book is bound in soft leather, and all of its page numbers are painted in gold. Unfortunately, after Schmerlin opened the old book, several loose pages fell out of it. After Schmerlin collected all the loose pages (25 two-sided sheets total), he decided to give himself some practice in math magic: he conjured a spell to add together all the page numbers on all the loose pages. The spell resulted in the number 2000. Schmerlin's wise owl claims that Schmerlin conjured an incorrect spell. How does the owl know? Let's look at the pair of page numbers that are written on the front and back sides of the same sheet of paper. Since these numbers differ by one, one of them is even and the other is odd. Hence, their sum is odd. Thus, the total sim of page numbers on the 25 torn sheets is equal to the sum of 25 odd numbers. This number has to be odd, and so cannot be 2000.

20. In the Land of Not-So-Far-Away there live 9 happy and 9 unhappy princesses. Schmerlin the Magician has just learned three new spells. The first spell makes any two unhappy princesses of his choice happy. His second spell transforms any pair of happy princesses into unhappy ones. The third spell switches the moods of a happy princess and an unhappy one: the happy princess becomes unhappy, and the unhappy one becomes happy.

    Schmerlin would like to make all the princesses happy. Prove that these three spells are not sufficient for his plan to come true:

    a. Explain what effect each of Schmerlin's spell has on the number of unhappy princesses. After a single spell, the number of unhappy princesses can go up by 2, down by 2, or remain the same.
    b. Currently, the number of unhappy princesses is odd. Suppose that Schmenlin utters one of his spells. Prove that the number of unhappy princesses will remain odd. The number of unhappy princesses remains odd since a single spell either does not change this number or changes it by 2.
    c. Suppose that Schmerlin performs several spells in a row. Prove that the number of unhappy princesses will remain odd. Since the number of unhappy princesses remains odd after each single spell, it will remain odd after several spells as well.
    d. Is it possible for the number of unhappy princesses to go down to zero eventually? If so, show how. If not, explain why. It is not possible because 0 is not an odd number.

21. Three types of magic fruit — apples of wisdom, pears of bravery, and plums of kindness — grow on the Magic Tree in the center of the Far Away Kingdom. From time to time, some of the fruits are harvested for

the benefit of the Kingdom. The Magic Tree immediately regrows the picked fruit according to the following set of rules:

a. If a single fruit is picked from the tree, another of the same kind grows in its place.
b. If 2 apples are picked, 4 pears grow back.
c. If 2 pears are picked, 4 plums grow back.
d. If 2 plums are picked, 4 apples grow back.
e. If 2 fruits of different kinds are picked, nothing else happens.

Currently, the tree has 11 apples, 10 pears and 8 plums. The wicked witch planes to weaken the Kingdom by stealing all the fruit. She intends to sneak to the tree several mornings in a row and pick one or two fruits every time. Is there a way for her to pick all the fruits of the tree? Either show how or explain why not. Originally, the tree has 29 fruits (an odd number). This number either remains the same, or goes up or down by 2. Therefore, whatever the witch does, the number of fruits will remain odd. Thus, 0 is not possible. (It is possible to reduce the tree to a single fruit of one type though. For example pick a pear and a plum 4 times, then pick an apple and a plum 4 times, then pick a pear and an apple 6 times. This leaves a single apple that will be replaced by a single apple every time it is picked.)

22. During the summer break, whenever Bella came to Tanya's house to play, she presented Tanya with a glass marble. Whenever Tanya came to came to Bella's house, she presented Bella with a glass marble as well. At the start of the summer, Bella owned 50 marbles. If it is known that Bella and Tanya had 35 play dates total, could it be possible that by the end of the summer Bella owned the same number of marbles –50? (Nobody else but Tanya presented Bella with marbles, and no marbles were bought or lost). Consider the number of marbles that Bella gave away and the number of marbles that Bella received during the summer. The sum of these two numbers is 35. Therefore, one of them is even and the other is odd — they cannot be equal.
Let's denote the first number by x, and the second by y. The number of of marbles that Bella owns at the end of the summer is equal to $50 - x + y$. Since x and y are different, this expression cannot be equal to 50.

23. Little Max had an odd number of quarters and an even number of dimes in his piggy bank. When he tried to calculate his wealth, he came up with the total of 3 dollars. Max's mom was pretty sure Max had made a mistake in his calculations. How did she know? The value of Max's quarters is an odd number of cents (because the product of two odds is odd). The value of Max's dimes is an even number of cents (because the product of two evens is even). The sum of odd and even is odd. Thus, the coins in the piggy bank are worth an odd number of cents. However, 3 dollars equal an even number of cents (300).

24. The integers from 1 to 18 are written on the board in a row. Can you insert plus and minus signs between them in such a way as to get an expression that is equal to 0? A total of 9 even and 9 odd digits are written on the board. Therefore, no matter how the plus and minus signs are inserted, the expression will have 9 even and 9 odd terms. Since the number of odd terms is odd, this expression will always evaluate to an odd number.