

BMC Worksheet: Odd and Even Numbers

1. There are several chairs in a rectangular room.
 - a. Can you arrange 4 chairs so as to have a chair next to each wall?
 - b. Can you do the same with 3 chairs?
 - c. Can you do the same with 2 chairs?
 - d. Can you arrange 12 chairs in such a way as to have 4 chairs next to each wall?
2. Imagine that you need to order a custom baking form for your cake. Since this is a custom order, you can choose any shape you want for the form.
 - a. Can you think of such an unusual shape that would make it possible to cut your cake into 4 pieces with one straight cut of a knife?
 - b. What would be the answer if the pieces had to be equal?
3. Two fathers and two sons had eggs for breakfast. Each person ate a whole egg, but there were only 3 eggs to begin with. How could this happen?
4. Two mountain trolls, Bob and Tob, were born exactly one year apart. Today they are celebrating their common birthday. Bob claims that their total age is 1128 years. However, Philip the hobbit is pretty sure that Bob is mistaken. How does he know?
5. In the country of Sugarland, King Donut announced a handsome reward to the first person able to divide 100 candies between 3 kids. Each child should get an odd number of candies and no candy should be left over. However, this reward has not yet been claimed. Explain why.
6. While visiting the Knights and Liars Island, you meet an islander. He says "Yesterday I baked 39 cookies for my party. Every adult at the party ate exactly 2 cookies, and every child ate 4 cookies. The cookies were a big success — by the end of the evening they'd all been eaten by the guests!" Is this islander a Knight or a Liar?
7. Slimy the Bog Witch invite the same number of witches and wizards to her birthday party. She bought 115 live frogs as party favors for her guests. She wants each guest to receive the same number of frogs. However, the witch's wise owl tells her that it is not possible to share the frogs this way. How does the owl know?
8. While visiting the Knights and Liars Island, Greg —the Math Circle Student— met an islander. Greg asked the islander how old he was. The islander replied, "If you multiply my age by 18, you get 1421." Greg immediately recognized that the islander was a liar — why?
9. You are cutting a loaf of bread with a knife. What is the maximum number of pieces of bread that you can get if you are allowed to make only three straight cuts? Every cut should be a through-cut. Also, you cannot stack or otherwise rearrange pieces of bread when cutting them.
10. Can a 5×5 chessboard be tiled by 2×1 dominoes? (To tile means to cover the board completely, without any overlaps.)

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11. How can you cut a round pizza with three straight knife cuts into 4, 5, 6, and 7 pieces? Every cut should be straight through; that is, it should start and end at the outer edge of the pizza.
12. Captain Cook kept a journal of his adventures. Once, on his way to the Cook Islands, he wrote, "Our flotilla has five ships. Each ship has an odd number of sailors on it. The total number of sailors in our expedition is 500." Mr. Hobbs, the history professor, claims that can't be right. Why?
13. In the morning, there were 5 spaceships at the spaceport of the planet Pandora. During the day, several more spaceships landed, and a few spaceships departed from the port. The ships always arrived in groups of 2 or 4, and departed in groups of 2. In the evening, the dispatcher counted 60 spaceships at the spaceport. Prove that the dispatcher miscalculated the number of spaceships.
14. There are three towns in a county: A, B, and C. The residents of town A never lie, those from town B never tell the truth, and those from town C alternate true and false statements.

One day, the county fire station received a call: "There is a fire in our town!" When the firefighter asked where the fire was, he got the reply, "In town B." Which town should the fire fighters go to? (Assume the call was from a resident of the town where the fire was.)
15. Peter says, "The day before yesterday I was 10, but next year I will turn 13." How can this be possible, if we know that Peter is not lying?
16. Johnny the Junior Hacker reprogrammed the elevator in the 100-story Boogle Corporation building: only two buttons are currently working. The first button sends the elevator 8 floors up, and the second one 6 floors down. (The elevator will not move if it is asked to go above the 100th floor or below the 1st floor.)
 - a. The company's CEO is currently drinking coffee on the first floor. (There is no lobby floor in the building.) Can he take the elevator to the 95th floor? If so, show how. If not, explain why.
 - b. Can he take the elevator to the 96th floor? If so, show how. If not, explain why.
17. Julia is walking home from school. She left the school 5 minutes earlier than her next-door neighbor, Josh. However, Josh is in a hurry because he wants to give Julia the cell phone that she left at school. Josh is walking 1.5 times faster than Julia. How soon will Julia get her cell phone back?
18. Bobby the Beaver is cutting several long logs into smaller pieces using a chainsaw (all cuts are across a log). Bobby made 30 cuts and ended up with 36 pieces of wood. How many long logs did he start with?
19. Schmerlin the Magician inherited a beautiful old book of spells. The book is bound in soft leather, and all of its page numbers are painted in gold. Unfortunately, after Schmerlin opened the old book, several loose pages fell out of it. After Schmerlin collected all the loose pages (25 two-sided sheets total), he decided to give himself some practice in math magic: he conjured a spell to add together all the page numbers on all the loose pages. The spell resulted in the number 2000. Schmerlin's wise owl claims that Schmerlin conjured an incorrect spell. How does the owl know?

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20. In the Land of Not-So-Far-Away there live 9 happy and 9 unhappy princesses. Schmerlin the Magician has just learned three new spells. The first spell makes any two unhappy princesses of his choice happy. His second spell transforms any pair of happy princesses into unhappy ones. The third spell switches the moods of a happy princess and an unhappy one: the happy princess becomes unhappy, and the unhappy one becomes happy.

Schmerlin would like to make all the princesses happy. Prove that these three spells are not sufficient for his plan to come true:

- a. Explain what effect each of Schmerlin's spell has on the number of unhappy princesses.
- b. Currently, the number of unhappy princesses is odd. Suppose that Schmenlin utters one of his spells. Prove that the number of unhappy princesses will remain odd.
- c. Suppose that Schmerlin performs several spells in a row. Prove that the number of unhappy princesses will remain odd.
- d. Is it possible for the number of unhappy princesses to go down to zero eventually? If so, show how. If not, explain why.

21. Three types of magic fruit — apples of wisdom, pears of bravery, and plums of kindness — grow on the Magic Tree in the center of the Far Away Kingdom. From time to time, some of the fruits are harvested for the benefit of the Kingdom. The Magic Tree immediately regrows the picked fruit according to the following set of rules:

- a. If a single fruit is picked from the tree, another of the same kind grows in its place.
- b. If 2 apples are picked, 4 pears grow back.
- c. If 2 pears are picked, 4 plums grow back.
- d. If 2 plums are picked, 4 apples grow back.
- e. If 2 fruits of different kinds are picked, nothing else happens.

Currently, the tree has 11 apples, 10 pears and 8 plums. The wicked witch plans to weaken the Kingdom by stealing all the fruit. She intends to sneak to the tree several mornings in a row and pick one or two fruits every time. Is there a way for her to pick all the fruits of the tree? Either show how or explain why not.

22. During the summer break, whenever Bella came to Tanya's house to play, she presented Tanya with a glass marble. Whenever Tanya came to Bella's house, she presented Bella with a glass marble as well. At the start of the summer, Bella owned 50 marbles. If it is known that Bella and Tanya had 35 play dates total, could it be possible that by the end of the summer Bella owned the same number of marbles -50 ? (Nobody else but Tanya presented Bella with marbles, and no marbles were bought or lost).

23. Little Max had an odd number of quarters and an even number of dimes in his piggy bank. When he tried to calculate his wealth, he came up with the total of 3 dollars. Max's mom was pretty sure Max had made a mistake in his calculations. How did she know?

24. The integers from 1 to 18 are written on the board in a row. Can you insert plus and minus signs between them in such a way as to get an expression that is equal to 0?

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Answers to BMC Worksheet: "Turning Lies into Truth" (Nov 3)

1. Suppose that a liar claims that his friend Joseph is telling the truth. What does this statement mean? *Joseph is a liar too, or Joseph is not his friend, or his friend is not Joseph, etc.*
2. If a liar declares that a certain marble is black, what does this mean? *The marble is not black.*
3. If a liar claims that there are no girls in the room, what would the true statement be? *There is at least one girl in the room.*
4. Suppose a liar tells you, "My grandfather is at least 100 years old." What would be the true age of the grandfather? *Less than 100 years old, i.e., 99, 98, 97,*
5. Suppose the Big Bad Wolf declares, "There are at least 10 pigs in this house." If you know that the wolf is a liar, what would be the true statement about the pigs in the house? *There are 9 or less pigs in the house.*
6. The Big Bad Wolf complains to you, "The pig I ate last night weighed less than 10 pounds." If you know that the wolf is lying about *the pig's weight*, what would be a true statement? (All pigs weigh an integer number of pounds.) *The pig weighed 10, 11, 12 or more pounds.*
7. Suppose a student claims that all students in his class are boys. If you know that this student is lying, what can you say about the number of boys in the class? *There is one or more girls in the class.*
8. * While visiting the Knights and Liars Island, you meet a boy who tells you that he is a liar. Does he live on the island or is he a tourist? *He is a tourist — neither a knight nor a liar would say they are a liar.*

**In the following problems (about Knights and Liars) you should try to do a TABLE !
Then reason through all the possibilities.**

9. * Two island boys, Sam and Bob, are introducing themselves to you. Sam says, "At least one of us is a liar." Can you figure out who is what? *Sam is a knight and Bob is a liar. If Sam were a liar then his statement would be true which would contradict the assumption that he was a liar. Therefore, he must be a knight, his statement must be true, and Bob must be the liar.*
10. * While visiting the Knights and Liars Island, I had a conversation with a local knight. I asked him the same question twice, and he gave me two different answers. What was my question? *A variety of answers are possible, for example, "What time is it?", "How many questions have I asked?" etc.*
11. * While visiting the island, you meet a group of three islanders: Tom, George and Betty. You ask each of them the same question: "How many knights are in your group?" Tom replies, "None." George says, "One." Can you figure out what Betty is? What is her response? *Tom cannot be telling the truth, since that would make him a knight, which contradicts the assumption that there is none; therefore, he is a liar. If there were two knights, George would have to be one of them. However, that would make George a liar also, which would contradict the assumption that there were two knights. Therefore, there must only be one knight and George is telling the truth. This makes Betty is a liar also, and she could give any number other than one.*

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12. * While visiting the Knights and Liars Island, you meet two islanders, James and Peter. James tells you that at least one of the two is a liar. Are James and Peter knights or liars? James is a knight and Peter is a liar. If James were a liar, then his statement would be true, which would contradict the assumption that he was a liar. Therefore, he must be a knight, his statement must be true, and Peter must be the liar.
13. Thirty children came to a party. Out of any 12 of them, at least one is a boy. Out of any 20 of them, at least one is a girl. How many boys and how many girls are there at the party? Make a diagram - it will be very helpful in this problem. Out of any 12 of them, at least one is a boy \rightarrow at most 11 girls; Out of any 20 of them, at least one is a girl \rightarrow at most 19 boys. If there were 10 or less girls, there would be 20 or more boys. If there are 18 or less boys, there are 12 or more girls. So there has to be exactly 11 girls and 19 boys since there are only 30 children total.
14. * While visiting the Knights and Liars Island, you come to a party. Every single person at this party tells you that there are some Liars in the room. What is really happening? How many Knights and Liars are at this party? (Remember that there can also be tourists, such as you. Tourists sometimes lie and sometimes tell the truth.) . If there *were* some liars in the room then those people would not say there are some liars since that would be the truth, therefore there must be no liars. Furthermore, there must be no knights since everyone said there were some liars, but we have just shown that there can be none. Therefore, everyone in the room must be a tourist.
15. * While visiting the Knights and Liars Island, you pass a beautiful garden where three islanders, Sam, Bob and Tom, are watching the sunset. You ask Sam, "Are you a Knight or a Liar?" Sam is shy; you cannot his quite answer. So you ask Bob, "What did Sam say?" Bob answers, that "He said that he is a Liar." "Don't trust Bob! Bob is a liar!" screams Tom. Can you decide whether Bob and Tom are Knights or Liars? First, we know that no liar would say he is a liar, since that would be the truth. Therefore, Bob must be a liar. Tom, then, is telling the truth and must be a knight.
16. * On the Island of Knights and Liars, Knights and Liars live in two separate villages. However, inhabitants of one village often visit their friends in the other village.
- This week, the island newspaper announced a contest: to come up with a single yes/no question that would allow a tourist to figure out whether he is in a knights' village or a liars' village. The question should be such that it can be addressed to anyone the tourist sees. It is not known in advance whether this local is a knight or a liar and whether he is a resident of this village or not. (There are no tourists other than the one asking the question.)
- Gregory, the Math Circle student, won the prize. What was his question? "Are you from this town?" If you are in a Knights' town, both knights and liars will say "yes". If you are in a Liars' town, both knights and liars will say "no".
- How would you modify the question if there were multiple villages of each type? "Are you from a town of this type?" If you are in a Knights' town, both knights and liars will say "yes". If you are in a Liars' town, both knights and liars will say "no".
17. Mother says to Max, "All fish love to swim." Max replies, "I love to swim; therefore, I am a fish." Is Max correct? Why or why not? No, other things beside fish can love to swim without making the statement "All fish love to swim" false. This is an error of logic, which is called "affirming the consequent". "If A then B"; "B therefore A" does not necessarily follow !

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18. Bim always tells the truth, and Bom always lies. What question could you ask each one of them if you want to get the same answer? (Assume that you don't know who does what.) "Do you always tell the truth?"
19. * Emma and Rachel are from the Island of Knights and Liars. One of them is a Liar and the other is a Knight. Emma claims that 2 rubies are more expensive than 3 sapphires. Rachel says that 3 rubies are more expensive than 4 sapphires. Is it the case that 12 rubies are more expensive than 18 sapphires?
Rachel says that $2R > 3S$ or (multiply both sides by 3): $6R > 9S$;
Emma says that $3R > 4S$ or (multiply both sides by 2): $6R > 8S$.
If Rachel is telling the truth, then so is Emma since $9S > 8S$, but if Rachel is lying, then Emma can still be telling the truth. Thus, Rachel is a liar, and Emma is a knight, or $8S < 6R < 9S$. Multiplying each term by 2, we get $16S < 12R < 18S$, hence, 12 rubies are NOT more expensive than 18 sapphires.
20. * Abby, Ben, Chris and Dan are all inhabitants of the Island of Knights and Liars. Abby claims that Ben is a liar. Dan states that Abby is a liar. Chris declares that both Abby and Ben are liars. Chris also states that Dan is a liar as well. Who is what? Justify your answer. Assume Abby is a knight, then Ben is a liar and Dan is a liar. But Chris claims that Abby is a liar, which, if our assumption is correct, means that Chris is a liar. However, Chris also claims that Ben and Dan are liars, which, if Abby is a knight, are both true statements. Since liars always lie, our original assumption that Abby is a knight must be incorrect, that is Abby is in fact a liar. Since Abby is a liar, Ben must be a knight. Also, Dan told the truth so he must be a knight. Chris claimed that BOTH Abby and Ben are liars, which is not true (only Abby is a liar), and he claims that Dan is a liar which is also not true. Therefore, Chris is a liar. Abby is a liar. Ben and Dan are knights.
21. The country of FarAwaynia is composed of several states: it also has several political parties. Once, a group of FarAwaynian politicians got together for a dinner. It is known that the group contained people from at least two different states and at least two different parties. Prove that there were at least two politicians at the dinner that both represented different states and belonged to different parties. There are 3 possibilities. First case: pick two politicians of different parties from different states. This meets the criteria. Second case: pick two politicians of different parties from the same state. Since we know that there must be another politician from a different state, we know that he must be from a different party than one of the two politicians from the first state. Again, this meets the criteria. Third case: pick two politicians of different states but of the same party. Since we know that there must be another politician from a different party, we know that he must be from a different state than at least one of the two politicians from the first party. This meets the criteria as well.
22. The weight of a gold bar is two-thirds of itself plus 5 pounds. What is the weight of the gold bar? Let x be the weight of the gold bar. Then $x = \frac{2}{3}x + 5$ pounds or $\frac{1}{3}x = 5$ pounds, therefore, $x = 15$ pounds.
23. A koala bear starts climbing up a eucalyptus tree that is 20 meters high. He starts at the bottom, climbing 5 meters up every day and sliding down 4 meters every night. How many days and nights will it take the koala to reach the top of the tree? The Koala will have gained 1 meter net increase at the end of each day and night cycle. After 15 days and nights, he will be at 15 meters. On the morning of the 16th day he is 5 meters below the top, he will obtain the top of the tree at the end of the day. It takes him 16 days and 15 nights to reach the top.