USAJMO Preparation

Elina Robeva

April 5, 2016

Problem 0.1 (Problem 5, USAJMO 2010)**.** Two permutations $a_1, \ldots, a_{2010}$ and $b_1, \ldots, b_{2010}$ of the numbers $1, 2, \ldots, 2010$ are said to *intersect* if $a_k = b_k$ for some value of k in the range $1 \leq k \leq 2010$.

(a). Show that there exist 2010 permutations of the numbers $1, 2, \ldots, 2010$ such that any other permutation is guaranteed to intersect at least one of these 2010 permutations;

(b). Show that there exist 1006 permutations of the numbers $1, 2, \ldots, 2010$ such that any other permutation is guaranteed to intersect at least one of these 1006 permutations.

Problem 0.2 (Problem 1, USAJMO 2011)**.** Find, with proof, all positive integers n for which $2^n + 12^n + 2011^n$ is a perfect square.

Hint: use mod 3 and mod 4.

Problem 0.3 (Problem 4, USAJMO 2011)**.** A *word* is defined as any finite string of letters. A word is a *palindrome* if it reads the same backwards as forwards. Let a sequence of words $W_0, W_1, W_2, \ldots$ be defined as follows: $W_0 = a, W_1 = b$, and for $n \geq 2$, W_n is the word defined by writing W_{n-2} followed by W_{n-1}. Prove that for any $n \geq 1$, the word formed by writing $W_1, W_2, \ldots, W_n$ in succession is a palindrome.

Problem 0.4 (Problem 2, USAJMO 2013)**.** Each cell of an $m \times n$ board is filled with some nonnegative integer. Two numbers in the filling are said to be *adjacent* if their cells share a common side. (Note that two numbers in cells that share only a corner are not adjacent). The filling is called a garden if it satisfies the following two conditions:

(i) The difference between any two adjacent numbers is either 0 or 1.

(ii) If a number is less than or equal to all of its adjacent numbers, then it is equal to 0 .

Determine the number of distinct gardens in terms of m and n .

Problem 0.5 (Problem 1, USAJMO 2014)**.** Let a, b, c be real numbers greater than or equal to 1. Prove that

$$\min\left(\frac{10a^2 - 5a + 1}{b^2 - 5b + 10}, \frac{10b^2 - 5b + 1}{c^2 - 5c + 10}, \frac{10c^2 - 5c + 1}{a^2 - 5a + 10}\right) \leq abc$$

Problem 0.6 (Problem 1, USAJMO 2015). Given a sequence of real numbers, a move consists of choosing two terms and replacing each with their arithmetic mean. Show that there exists a sequence of 2015 distinct real numbers such that after one initial move is applied to the sequence – no matter what move – there is always a way to continue with a finite sequence of moves so as to obtain in the end a constant sequence.

Problem 0.7 (Problem 2, USAJMO 2015). Solve in integers the equation

$$x^2 + xy + y^2 = \left(\frac{x+y}{3} + 1\right)^3.$$

Problem 0.8 (Problem 4, USAJMO 2015). Find all functions $f : \mathbb{Q} \to \mathbb{Q}$ such that

$$f(x) + f(t) = f(y) + f(z)$$

for all rational numbers $x < y < z < t$ that form an arithmetic progression. ($\mathbb{Q}$ is the set of all rational numbers.)

Problem 0.9. (a). Find all sequences a_n, $n = 0, 1, 2, \ldots$, such that

$$a_n = 2a_{n-1}, \text{ for all } n \geq 1.$$

(b). Find all sequences a_n, $n = 0, 1, 2, \ldots$, such that

$$a_n = a_{n-1} + 2a_{n-2}, \text{ for all } n \geq 2.$$

(c). Find all sequences a_n, $n = 0, 1, 2, \ldots$, such that

$$a_n = 4a_{n-1} - 4a_{n-2}, \text{ for all } n \geq 2.$$

(d). Find all sequences a_n, $n = 0, 1, 2, \ldots$, such that

$$a_n = a_{n-1} + 2a_{n-2} + 6, \text{ for all } n \geq 2.$$