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RANDOM WALKS

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Warm-ups: Recursive Probability and Expected Value

1 Two players alternately toss a penny, and the one that first tosses heads wins. What is the probability that

(a) the game never ends?

(b) the first player wins?

(c) the second player wins?

2 It costs a consumer \$1 to buy a Klopstockia lottery ticket. The buyer then scratches the ticket to see the prize. Compute, to the nearest penny, the expected profit that the state of Klopstockia makes per ticket sold, given the following scenarios for prizes awarded. (The state will make a profit if the expected value of the lottery ticket is *less* than \$1.)

(a)

Prize	\$1	\$10
Probability	$\frac{1}{10}$	$\frac{1}{1{,}000}$

(b)

Prize	\$1	\$10	a free lottery ticket
Probability	$\frac{1}{10}$	$\frac{1}{1{,}000}$	$\frac{1}{5}$

3 (a) On average, how many times must a die be thrown until one gets a 6?

(b) How many times, on average, should one toss a fair die in order to see all 6 possible outcomes?

4 *A Bug on a Cube*. Imagine a bug that crawls along the edges of a cube. The bug does not change directions while traveling on an edge. Two adjacent vertices, F and P, have food and poison, respectively. If the bug reaches either of these vertices, it stops traveling. Whenever the bug reaches one of the other six vertices, it has a choice of three edges on which to travel and it chooses randomly (i.e., with probability of $1/3$ for each choice). For each of these six starting vertices, compute the probability that the bug lives (i.e., reaches F before reaching P).

5 *A true story*. When I first organized the Bay Area Mathematical Olympiad, I needed to send registration forms out with random ID numbers for participants. So I made a list of the numbers from 1 to 1000, and then used my sampling software to take a random sample of size 1000 from these numbers. However, I stupidly forgot to check the "sample without replacement" button and instead I sampled *with* replacement. How many distinct ID numbers were produced?

6 *A true story*. In the SF Math Circle for elementary school kids, 11 8-year-old kids stood in a circle. They wrote their names on a piece of paper, and the instructor put them in a box and shook the box. Then each kid randomly chose a name. The instructor handed a kid a beanbag ball and the kid then tossed the ball to the person whose name they had. And so it continued. If not all kids got a ball tossed to them, the instructor gave the ball to one of those left-out kids and the process continued.

(a) If a kid ended up tossing a ball to him or herself, that kid cried. On average, how many kids cry?

(b) What is the probability that no kids cry?

(c) The instructor wanted all the kids to be able to toss the ball without intervention. In other words, ideally, all 11 kids will form a ”cycle.” What is the probability that this happens?

(d) If all the kids are not in one cycle, the instructor asked the kids to change names so that this can be achieved. To keep anarchy at bay, the instructor only allowed two kids at a time to exchange their slips of paper. On average, how many such exchanges are needed?

(e) Another desirable scenario for the instructor was for a majority of the kids to be in a cycle. Otherwise, kids have tantrums. What’s the probability of a tantrum?

7 *Snake necklaces*. Imagine a pit of 100 snakes, and James Bond is thrown into the pit. He fearlessly, and randomly, grabs ends of snakes (ignoring whether it is the head or tail) and deftly ties them together. He keeps doing this until he is left with a bunch of ”snake necklaces” that cannot harm him. How many necklaces will there be?

8 *A Gambling “System.”* Suppose you are playing a game with a 50% chance of winning each time. You can bet any amount, and if you win, you win twice your bet. If you lose, you lose your bet. In other words, if you bet B dollars, your *profit* is $\pm B$ depending on whether you win or lose. You decide that you will play, stopping as soon as you win, doubling the size of your bet each time. You are guaranteed to make a profit? Right? Use expectation to show that this won’t work. What if you triple instead of double?

9 *The St. Petersburg Paradox*. Consider the following game. I will flip a fair coin until it shows up heads. We keep track of the number of flips until this happens. If it happens on the first flip, I’ll pay you \$2. If it takes two flips, then I’ll pay you \$4. Three flips, \$8, etc. In other words, if it takes n flips until the first head, I will pay you 2^n dollars. Pretty sweet game!

How much is this game worth *to you*? In other words, if a there was a ticket that allowed you to play the game once with me (I flip the coin until it is heads, and pay you the appropriate amount), how much would you pay for the ticket? Clearly, you’d pay at least 1 dollar. In fact, you’d almost certainly pay 2 dollars. How about 3? 4? 5? More?

Random Walks

10 *The Classic Gambler's Ruin Problem.* Two players take turns tossing a fair coin. If the coin is heads, player A gives player B a dollar. If the coin turns up tails, B gives A a dollar. Player A starts with a dollars, and player B starts with b dollars (a and b are non-negative integers). Once a player goes bankrupt (i.e., has zero dollars) the game is over. What is the probability that A goes bankrupt?

What happens if the probabilities are not equal; i.e., what if the probability that the coin is heads is p, for some fixed $0 \le p \le 1$.

11 *What a Loser!* You arrive in Las Vegas with \$100 and decide to play roulette, making the same bet each time, until you are either bankrupt or have doubled your money. Which of the following strategies is best? Or are they all equally bad?

(a) Making bets of \$1 each time.

(b) Making bets of \$10 each time.

(c) Making a single bet of \$100.

12 Prove that

$$\frac{1}{2\sqrt{n}} < \frac{\binom{2n}{n}}{2^{2n}} < \frac{1}{\sqrt{2n}}.$$

13 In a fair ($p = q = 1/2$) random walk on the number line from 0 to N with boundaries at 0 and N, define w_k to be the probability of "winning" (reaching N before reaching 0), starting at k.

(a) Find a recurrence formula for w_k.

(b) Solve it!

14 In a *unfair* random walk (at each turn, the probability is p that you move one step to the right and $q = 1 - p$ that you go one step to the left) on the number line from 0 to N with boundaries at 0 and N, define w_k to be the probability of "winning" (reaching N before reaching 0), starting at k.

(a) Find a recurrence formula for w_k.

(b) Solve it!

15 In a fair ($p = q = 1/2$) random walk on the number line from 0 to N with boundaries at 0 and N, define the random variable s_k be the number of steps it takes to get to 0, starting at k.

(a) Find a recurrence formula for $e_k := E(s_k)$ and solve it. It should be a quadratic function equalling zero at $k = 0$ and $k = N$.

16 Answer the above question for the unfair random walk.

17 Consider the random walk in d dimensions starting at $\mathbf{0}$, where with probability $1/2d$, you can move ± 1 in one of the d directions.

(a) Define the random variable L_n to be the distance from the origin that you are at step n. Can you find $E(L_n)$? If that's too hard, how about using problem #15 above, and/or contemplating $E(L_n^2)$.

(b) Define R to be the number of times you return to the origin. What is $E(R)$?

(c) What is the probability that you return to the origin? Does it depend on d?

Miscellaneous Challenging Problems

18 A "continuous" roulette wheel has all real numbers from 0 to 1. Repeatedly spin the wheel until the numbers that arise add up to at least 1. What is the expected number of spins?

19 *A Problem from The 2000 Bay Area Math Meet (BAMM).* Consider the following experiment:

- First a random number p between 0 and 1 is chosen by spinning an arrow around a dial which is marked from 0 to 1. (This way, the random number is "uniformly distributed"—the chance that p lies in the interval, say, from 0.45 to 0.46 is exactly $1/100$; and the chance that p lies in the interval from 0.324 to 0.335 is exactly $11/1000$, etc.)
- Then an unfair coin is built so that it lands "heads up" with probability p.
- This coin is then flipped 2000 times, and the number of heads seen is recorded.

What is the probability that exactly 1000 heads were recorded?

20 *A Problem from BAMM 1999.* Eleven points are chosen randomly on the surface of a sphere. What is the probability that all eleven points lie on some hemisphere of this sphere?