AMC 10 Contest Problems

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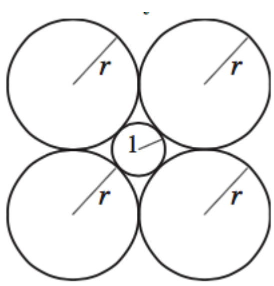
Today's session is dedicated to exploring a variety of AMC 10 contest problems you might find. So, if you get bored, wait a few minutes, and you might encounter something new and inspiring!

Remark. Some AMC 10 contest guidelines:

- The AMC 10 is a twenty-five question, multiple choice test. You have choices A, B, C, D, and E. Only one is correct. You have 75 MINUTES working time to complete the test. No calculators are allowed.
- SCORING: You will receive 6 points for each correct answer, 1.5 points for each problem left unanswered, and 0 points for each incorrect answer.
- How many questions should you answer on AMC 10 and AMC 12 to get the best chances of qualifying for AIME? The optimal strategy is to answer 21 questions on AMC 10 (if you make two mistakes, you still qualify with a score of 120). You can afford to make two mistakes and still achieve a qualifying score of 120 ($19 \times 6 + 4 \times 1.5 = 120$). This strategy allows you to spend about 3.5 minutes on each of the 21 questions you will answer and skip the four most difficult questions (usually questions #22–#25).

1 Warm-Up Exercises

1. (*AMC 10A 2013*) Central High School is competing against Northern High School in a backgammon match. Each school has three players, and the contest rules require that each player play two games against each of the other school's players. The match takes place in six rounds, with three games played simultaneously in each round. In how many different ways can the match be scheduled?
2. (*AMC 10 2001*) The median of the list $n, n+3, n+4, n+5, n+6, n+8, n+10, n+12, n+15$ is 10. What is the mean?
3. (*AMC 10 2001*) What is the maximum number for the possible points of intersection of a circle and a triangle?
4. (*AMC 10 2007*) A circle of radius 1 is surrounded by 4 circles of radius r as shown. What is r?



5. (*AMC 10 2007*) All sides of the convex pentagon $ABCDE$ are of equal length, and $\angle A = \angle B = 90^\circ$. What is the degree measure of $\angle E$?

6. A pyramid with a square base is cut by a plane that is parallel to its base and is 2 units from the base. The surface area of the smaller pyramid that is cut from the top is half the surface area of the original pyramid. What is the altitude of the original pyramid?

2 General Contest Strategies

- Take advantage of symmetry.
- Know what kind of problem you're looking at (Algebra, Number Theory, Geometry, Counting & Probability, General Logic / Problem-Solving).
- Practice. All past AMC 10 contests are available online.

2.1 Number Theory

Example. *Sum of the Factors:*

What is the sum of the factors of 20? Find the prime factorization:

$$20 = 2^2 \cdot 5$$

Notice what happens when we distribute the sum of the powers of 2 to the sum of the powers of 5:

$$(2^0+2^1+2^2)(5^0+5^1) = (1+2+4)(1+5) = 1(1+5)+2(1+5)+4(1+5) = (1+2+4)(1+5) = 42.$$

Distribution gives us every factor!

Example. *Find the sum of the factors of* (5!).

Solution: Begin with the prime factorization of (5!):

$$5! = 5 \cdot \ 4 \cdot 3 \cdot 2 \cdot 1 = 2^3 \cdot 3 \cdot 5.$$

The sum of the factors is $(1 + 2 + 4 + 8)(1 + 3)(1 + 5) = 360$.

2.2 Exercises

1. **Product of the Factors.** Now that you can count the number of factors quickly, let's find a quick way to find the product of these factors. Make a list of all factors of 96 and 196. Be sure to list all the factors as pairs! How many factors does 96 have? How many pairs is this? Try to come up with a general formula for the product of the factors.

2. (*2003 AIME*) Two positive integers differ by 60. The sum of their square roots is the square root of an integer that is not a perfect square. What is the maximum possible sum of the two integers?

3. There are 9 divisors for number A and 10 divisors for number B. The least common multiple of A and B is 2800. What are these two numbers?

4. (*2007 AMC 10*) How many pairs of positive integers (a, b) are there such that a and b have no common factors greater than 1 and

$$\frac{a}{b} + \frac{14b}{9a}$$

is an integer? ?????

3 Counting and Probability

3.1 Exercises

1. (*2006 AMC 10B*) Bob and Alice each have a bag that contains one ball of each of the colors blue, green, orange, red, and violet. Alice randomly selects one ball from her bag and puts it into Bob's bag. Bob then randomly selects one ball from his bag and puts it into Alice's bag. What is the probability that after this process the contents of the two bags are the same?

2. (*2006 AMC 10B*) For a particular peculiar pair of dice, the probabilities of rolling 1, 2, 3, 4, 5, and 6, on each die are in the ratio $1 : 2 : 3 : 4 : 5 : 6$. What is the probability of rolling a total of 7 on the two dice?

3. (*2006 AMC 10A*) How many four-digit positive integers have at least one digit that is a 2 or a 3?

4. (*2009 AMC*) Two cubical dice each have removable numbers 1 through 6. The twelve numbers on the two dice are removed, put into a bag, then drawn one at a time and randomly reattached to the faces of the cubes, one number to each face. The dice are then rolled and the numbers on the two top faces are added. What is the probability that the sum is 7?

4 Geometry

Remark. There are often different types of geometry problems, including: 3D Geometry, Area, Circles, Similarity, Triangles, Parallel Lines, Polygons, Power of a Point, Quadrilaterals, and Triangle Inequality.

4.1 Exercises

1. Points A, B, C and D lie on a line, in that order, with $AB = CD$ and $BC = 12$. Point E is not on the line, and $BE = CE = 10$. The perimeter of $\triangle AED$ is twice the perimeter of $\triangle BEC$. Find AB.

2. (*2003 AMC10A*) The number of inches in the perimeter of an equilateral triangle equals the number of square inches in the area of its circumscribed circle. What is the radius, in inches, of the circle?

3. (*2006 AMC10B*) In a triangle with integer side lengths, one side is three times as long as a second side, and the length of the third side is 15. What is the greatest possible perimeter of the triangle?

5 Algebra

1. (*2006 AMC12A*) Sandwiches at Joe's Fast Food cost \$3 each and sodas cost \$2 each. How many dollars will it cost to purchase 5 sandwiches and 8 sodas?

2. Real number x satisfies $x + 1 = 5$. Find $x^3 + 1$.

3. If $t + u = 1$ and $t^2 + u^2 = 3$, compute $t^4 + u^4$.

4. Is $2017^4 + 4^2017$ a prime number? Explain.

6 Challenge of the Day

Denote $m \circ n = \frac{mn+4}{m+n}$. Find the value of

$$((((2016 \circ 2015) \circ 2014) \circ \cdots \circ 2) \circ 1) \circ 0.$$