AROUND $(\mathbf{1}+\sqrt{\mathbf{2}})^{\mathbf{n}}$

1. Calculator tricks. Let us take a simple pocket calculator and compute 1-st, 2-nd, 3-rd, and so on, powers of the number $1+\sqrt{2}$. Here are the results:

$$\begin{array}{rcr}
1+\sqrt{2} & = & 2.414213\ldots \\
(1+\sqrt{2})^2 & = & 5.828427\ldots \\
(1+\sqrt{2})^3 & = & 14.071067\ldots \\
(1+\sqrt{2})^4 & = & 33.970562\ldots \\
(1+\sqrt{2})^5 & = & 82.012193\ldots \\
(1+\sqrt{2})^6 & = & 197.994949\ldots \\
(1+\sqrt{2})^7 & = & 478.002092\ldots \\
(1+\sqrt{2})^8 & = & 1153.999133\ldots \\
(1+\sqrt{2})^9 & = & 2786.000359\ldots \\
(1+\sqrt{2})^{10} & = & 6725.999851\ldots
\end{array}$$

We see that these numbers become closer and closer to integers. Also, we can observe an alternating: first a number is slightly less than an integer, then slightly more than an integer, then again less, then again more, and so on. WHY?

Before answering this question, let us look at a different calculation. Take the same powers of $1+\sqrt{2}$ and divide them by $\sqrt{2}$. We observe the following:

$$\begin{array}{rcr}
(1+\sqrt{2})/\sqrt{2} & = & 1.707106\ldots \\
(1+\sqrt{2})^2/\sqrt{2} & = & 4.121320\ldots \\
(1+\sqrt{2})^3/\sqrt{2} & = & 9.949747\ldots \\
(1+\sqrt{2})^4/\sqrt{2} & = & 24.020815\ldots \\
(1+\sqrt{2})^5/\sqrt{2} & = & 57.991378\ldots \\
(1+\sqrt{2})^6/\sqrt{2} & = & 140.003571\ldots \\
(1+\sqrt{2})^7/\sqrt{2} & = & 337.998521\ldots \\
(1+\sqrt{2})^8/\sqrt{2} & = & 816.000613\ldots \\
(1+\sqrt{2})^9/\sqrt{2} & = & 1969.999746\ldots \\
(1+\sqrt{2})^{10}/\sqrt{2} & = & 4756.000105\ldots
\end{array}$$

We observe the same close-to-integer pattern. Again, WHY? The explanation is very simple.

2. Here is WHY. We all know that there is a formula for $(1+x)^n$, the so called Newton's binomial formula. We, actually, do not need this much. All we need, is the fact that $(1+x)^n$ is a degree n polynomial with integer coefficients. For example,

$$(1+x)^5 = 1+5x+10x^2+10x^3+5x^4+x^5.$$

Plug into this formula $x=\sqrt{2}$ and then $x=-\sqrt{2}$ (using the fact $(\sqrt{2})^2=2$, $(\sqrt{2})^3 = 2\sqrt{2}$, $(\sqrt{2})^4=4$, $(\sqrt{2})^5=4\sqrt{2}$):

$$(1+\sqrt{2})^5 = 1+5\sqrt{2}+20+20\sqrt{2}+20+4\sqrt{2},$$
$$(1-\sqrt{2})^5 = 1-5\sqrt{2}+20-20\sqrt{2}+20-4\sqrt{2}.$$

Add up these two equalities. All the square roots cancel, and we get:

$$(1+\sqrt{2})^5 + (1-\sqrt{2})^5 = 82, \text{ an integer.}$$

But a similar argumentation can be applied to $(1 \pm \sqrt{2})^n$ with an arbitrary positive integral exponent n. We see that the sum $(1+\sqrt{2})^n + (1-\sqrt{2})^n$ is always an integer. But $\left|1-\sqrt{2}\right| < 1$; hence $(1-\sqrt{2})^n$ is very close to zero. Moreover, the sign of $(1-\sqrt{2})^n$ is alternating: $-, +, -, +, \ldots$. From this we deduce that $(1+\sqrt{2})^n$ is almost an integer, precisely as we observed before; and the alternating pattern described above is also confirmed.

Now let us turn to the division by $\sqrt{2}$:

$$\begin{aligned}\frac{(1+\sqrt{2})^5}{\sqrt{2}} &= \frac{1}{\sqrt{2}} + 5 + 20\sqrt{2} + 20 + \frac{20}{\sqrt{2}} + 4,\\ \frac{(1-\sqrt{2})^5}{\sqrt{2}} &= \frac{1}{\sqrt{2}} - 5 + 20\sqrt{2} - 20 + \frac{20}{\sqrt{2}} - 4.\end{aligned}$$

All is as before, only instead of the addition, we do the subtraction:

$$\frac{(1+\sqrt{2})^5}{\sqrt{2}} - \frac{(1-\sqrt{2})^5}{\sqrt{2}} = 58, \text{ an integer.}$$

We do the same for an arbitrary n and conclude that $\dfrac{(1+\sqrt{2})^n}{\sqrt{2}} - \dfrac{(1-\sqrt{2})^n}{\sqrt{2}}$ is always an integer, so $\dfrac{(1+\sqrt{2})^n}{\sqrt{2}}$ is almost an integer.

Let us notice that these arguments can be applied not only to $1+\sqrt{2}$. For example, the numbers $(1+\sqrt{3})^n$ and $\dfrac{(1+\sqrt{3})^n}{\sqrt{3}}$ will again be "almost integers" (although, since $\sqrt{3}-1$ is not so close to zero as $\sqrt{2}-1$, the difference between $1+\sqrt{3}$ and the nearest integer will tend to zero not so fast as the similar difference for $1+\sqrt{2}$). In the same time, $(1+\sqrt{5})^n$ is not expected to be close to an integer, while $(2+\sqrt{5})^n$ is very close to an integer. Here are some computations confirming these observations:

$$\begin{aligned}(1+\sqrt{3})^{10} &= 23167.955801\ldots\\ (1+\sqrt{5})^{10} &= 125943.674246\ldots\\ (2+\sqrt{5})^{10} &= 1860497.999999463\ldots\\ \frac{(1+\sqrt{3})^{10}}{\sqrt{3}} &= 13376.025518\ldots\\ \frac{(2+\sqrt{5})^{10}}{\sqrt{5}} &= 832040.00000024\ldots\end{aligned}$$

3. What are these integers? Thus, the numbers $(1+\sqrt{2})^n$ and $\dfrac{(1+\sqrt{2})^n}{\sqrt{2}}$ (and some other similar numbers) are very close to integers. Then it is permissible to ask, TO WHAT INTEGERS are they close? Let us have a look at these integers.

$n =$	1	2	3	4	5	6	7	8	9	10
$(1+\sqrt{2})^n \approx$	2	6	14	34	82	198	478	1154	2786	6726
$(1+\sqrt{2})^n/\sqrt{2} \approx$	2	4	10	24	58	140	338	816	1970	4756

What can be said about these integers? Well, they are all even, but it is not a big deal: there are so many even integers! A further observation reveals a more interesting property: in the both sequences, every number (starting with the third one) is equal to the twice preceding number plus the next preceding number! Look: $14 = 2 \cdot 6 + 2$, $34 = 2 \cdot 14 + 6$, $82 = 2 \cdot 34 + 14$, etc. And also $10 = 2 \cdot 4 + 2$, $24 = 2 \cdot 10 + 4$, $58 = 2 \cdot 24 + 10$, etc. Certainly, our observation is experimental, it is not a proof. But it is not hard to prove it. Let us do this.

Actually not only the integers a_n which approximate $b_n = (1+\sqrt{2})^n$, satisfy the equation

$$a_{n+2} = 2a_{n+1} + a_n,$$

but the numbers b_n themselves satisfy the equation

$$b_{n+2} = 2b_{n+1} + b_n.$$

Indeed,

$$\begin{aligned} b_{n+2} &= (1+\sqrt{2})^{n+2} = (1+\sqrt{2})^2(1+\sqrt{2})^n = (3+2\sqrt{2})(1+\sqrt{2})^n \\ &= (2(1+\sqrt{2})+1)(1+\sqrt{2})^n = 2(1+\sqrt{2})(1+\sqrt{2})^n + (1+\sqrt{2})^n = 2b_{n+1} + b_n. \end{aligned}$$

Turning to a_n, we notice that since all the numbers a_n are integers, the difference $a_{n+2} - (2a_{n+1} + a_n)$ must be an integer. But it is very close to $b_{n+2} - (2b_{n+1} + b_n)$ which is zero. Hence, $a_{n+2} - (2a_{n+1} + a_n)$ must be zero as well.

For the numbers $(1+\sqrt{2})^n/\sqrt{2}$ we do not need any additional work: the division by $\sqrt{2}$ will not spoil the identity $b_{n+2} = 2b_{n+1} + b_n$; thus the second sequence of integers has the same $a_{n+2} = 2a_{n+1} + a_n$ property.

We can say that the both sequences are similar to the Fibonacci sequence, with the only difference arising from the coefficient 2 at a_{n+1}. So, they are some sort of "super-Fibonacci numbers."

4. A formula for super-Fibonacci. Consider a sequence

$$a_0, a_1, a_2, a_3, \ldots$$

of – what? Well, we can say, of integers, but we can assume they being real, or even posibly complex numbers. Suppose that for every n we have $a_{n+2} = K \cdot a_{n+1} + L \cdot a_n$ with some fixed K and L. For example, for the classical Fibonacci sequence, we have $K = L = 1$, while for the sequences considered above $K = 2$ and $L = 1$. (Actually, the sequences above started with a_1; to supplement them with a term a_0, we must have for the first sequence $6 = 2 \cdot 2 + a_0$, so $a_0 = 6 - 2 \cdot 2 = 2$ and for the second sequence $4 = 2 \cdot 2 + a_0$, so $a_0 = 0$. By the way, in the classical Fibonacci sequence, $a_1 = a_2 = 1$; so, we must have $a_0 + 1 = 1 \Rightarrow a_0 = 0$.)

For members a_n of our sequence we have a recursive formula which makes it possible to calculate all a_n's provided that we know a_0 and a_1. But how to write a direct formula which allows to find a value of a_n without knowing values of all the previous terms of the sequence? This is what we are going to do now.

Our condition for a_n may be regarded as an infinite sequence of equations,

$$\begin{cases} a_2 = Ka_1 + La_0 \\ a_3 = Ka_2 + La_1 \\ a_4 = Ka_3 + La_2 \\ \dots\dots\dots\dots\dots\dots \end{cases} \tag{1}$$

We know that this system has infinitely many solutions: choose arbitrary values for a_0 and a_1, and the equations will successively provide values for a_2, a_3, a_4 and so on. Mathematicians, when they deal with a problem with multiple solution, often start with an attempt to find solution in some particular, convenient form. Using this trick, we will try to find solutions of the infinite system (1) in the following form: $a_n = \lambda^n$ where λ is some (unknown) number different from 0.

The system (1) becomes

$$\begin{cases} \lambda^2 = K\lambda + L \\ \lambda^3 = K\lambda^2 + L\lambda \\ \lambda^4 = K\lambda^3 + L\lambda^2 \\ \dots\dots\dots\dots\dots\dots \end{cases}$$

Since $\lambda \neq 0$, all these equations are the same; so, the numbers $a_n = \lambda^n$ satisfy our condition if and only if λ is one of the solutions of the quadratic equation $\lambda^2 - K\lambda - L = 0$.

Suppose that this quadratic equation has two real solutions, λ_1 and λ_2. Then we have at least two solutions of the system (1): $a_n = \lambda_1^n$ and $a_n = \lambda_2^n$. But $a_n = A\lambda_1^n$ with an arbitrary real A is also a solution: if $a_{n+2} = Ka_{n+1} + La_n$, then $Aa_{n+2} = KAa_{n+1} + LAa_n$ as well. Equally well, $a_n = B\lambda_2^n$ with an arbitrary real B is a solution. Finally, the "sum" of two solutions is a solution, so

$$a_n = A\lambda_1^n + B\lambda_2^n \tag{2}$$

is a solution of the system (1) for arbitrary real A and B. Let us observe now that there are no other solutions.

Indeed, a solution is fully determined by an arbitrary choice of a_0 and a_1. For given a_0 and a_1, we want to find A and B such that

$$\begin{cases} a_0 = A\lambda_1^0 + B\lambda_2^0 = A + B. \\ a_1 = A\lambda_1^1 + B\lambda_2^1 = A\lambda_1 + B\lambda_2. \end{cases}$$

The solution of this system of equations (with the unknowns A and B) is

$$A = \frac{a_0\lambda_2 - a_1}{\lambda_2 - \lambda_1}, \ B = \frac{a_0\lambda_1 - a_1}{\lambda_1 - \lambda_2}. \tag{3}$$

Thus, the formulas (2) and (3) (with λ_1 and λ_2 being the solutions of the equation $\lambda^2 - K\lambda - L = 0$) provide all the sequences $\{a_n\}$ with the properties required.

5. Examples old and new. Let us begin with the two sequences considered in the beginning of this note (Section 1). For the first sequence, $2, 2, 6, 14, 34, 82, \ldots$, we have: $K = 2, L = 1, a_0 = 2, a_1 = 2$. The solutions of the equation $\lambda^2 - 2\lambda - 1 = 0$ are $\lambda_1 = 1 + \sqrt{2}$ and $\lambda_2 = 1 - \sqrt{2}$, and formulas (3) give

$$A = \frac{2(1-\sqrt{2}) - 2}{(1-\sqrt{2}) - (1+\sqrt{2})} = 1 \text{ and } B = \frac{2(1+\sqrt{2}) - 2}{(1+\sqrt{2}) - (1-\sqrt{2})} = 1.$$

Thus,

$$a_n = (1+\sqrt{2})^n + (1-\sqrt{2})^n$$

which we know only too well. For the second sequence, $0, 2, 4, 10, 24, 58, \ldots$, all the numbers $K, L, a_0, a_1, \lambda_1$, and λ_2 are the same as before, with only one exception: a_0 is 0, not 2. In this case, formulas (3) give

$$A = \frac{-2}{(1-\sqrt{2}) - (1+\sqrt{2})} = \frac{-2}{-2\sqrt{2}} = \frac{1}{\sqrt{2}} \text{ and } B = \frac{-2}{(1+\sqrt{2}) - (1-\sqrt{2})} = \frac{-2}{2\sqrt{2}} = -\frac{1}{\sqrt{2}}.$$

Thus,

$$a_n = \frac{(1+\sqrt{2})^n}{\sqrt{2}} - \frac{(1-\sqrt{2})^n}{\sqrt{2}}$$

which we also know.

Next, let us consider the classical Fibonacci sequence, $0, 1, 1, 2, 3, 5, 8, 13, 21, 34, \ldots$. In this case, $K = 1, L = 1, a_0 = 0, a_1 = 1$, the equation $\lambda^2 - \lambda - 1 = 0$ has solutions $\lambda_1 = \dfrac{1+\sqrt{5}}{2}$ (the "golden ratio") and $\lambda_2 = \dfrac{1-\sqrt{5}}{2}$. By formulas (3),

$$A = \frac{-1}{\dfrac{1-\sqrt{5}}{2} - \dfrac{1+\sqrt{5}}{2}} = \frac{1}{\sqrt{5}}, \ B = \frac{-1}{\dfrac{1+\sqrt{5}}{2} - \dfrac{1-\sqrt{5}}{2}} = -\frac{1}{\sqrt{5}},$$

whence

$$a_n = \frac{1}{\sqrt{5}} \cdot \left(\left(\frac{1+\sqrt{5}}{2} \right)^n - \left(\frac{1-\sqrt{5}}{2} \right)^n \right).$$

This formula for the Fibonacci numbers is very popular: it appears in hundreds of textbooks. We only notice that it may seem amazing that an expression involving so many square roots of 5 turns out to be an integer.

Finally, let us notice that the solutions of the quadratic equation $\lambda^2 - K\lambda - L = 0$ are not necessarily irrational: they may be integers. For example, let $a_{n+2} = a_{n+1} + 6a_n$. We will consider two such sequences: with $a_0 = 0, a_1 = 1$ and with $a_0 = 1, a_1 = 0$:

$$0, 1, 1, 7, 13, 55, 133, \ldots,$$
$$1, 0, 6, 6, 42, 78, 798, \ldots.$$

The solutions of the quadratic equation $\lambda^2 - \lambda - 6 = 0$ are $\lambda_1 = 3$ and $\lambda_2 = -2$. For the first sequence, formulas (3) give

$$A = \frac{-1}{-2-3} = \frac{1}{5}, \; B = \frac{-1}{3+2} = -\frac{1}{5},$$

whence

$$a_n = \frac{1}{5}\left(3^n - (-2)^n\right) = \frac{1}{5}\left(3^n + (-1)^{n-1}2^n\right).$$

For the second sequence,

$$A = \frac{-2}{-2-3} = \frac{2}{5}, \; B = \frac{3}{3+2} = \frac{3}{5},$$

and

$$a_n = \frac{2}{5} \cdot 3^n + \frac{3}{5} \cdot (-2)^n = \frac{6}{5}\left(3^{n-1} + (-1)^n 2^{n-1}\right).$$

6. The case of complex roots. If all the numbers K, L, a_n are complex, then the equation $\lambda^2 - K\lambda - L = 0$ has two complex solutions, and (in the case when they are different) we can repeat all said in Section 4 without any changes (having in mind that all the numbers we deal with are complex).

It becomes more interesting, if all K, L, a_n are real, but the solutions of the equation $\lambda^2 - K\lambda - L = 0$ are not real. For example, let $a_{n+2} = 2a_{n+1} - 3a_n$ and $a_0 = 0$, $a_1 = 1$. The solutions of the equation $\lambda^2 - 2\lambda + 3 = 0$ are not real: $\lambda_{1,2} = 1 \pm i\sqrt{2}$. But the sequence is real:

$$0, 1, 2, 1, -4, -11, -10, 13, 56, 73, -22, -263, -460, -131, 1118, 2629, 1904, -4079, \ldots$$

How to find a formula for a general term of a sequence like this? Since the whole sequence is real, we want to have a pure real formula, without any explicit usage of complex numbers.

So, let $a_{n+2} = K \cdot a_{n+1} + L \cdot a_n$ and K, L, a_0, a_1 are real numbers, but the polynomial $\lambda^2 - K \cdot \lambda - L = 0$ has non-real roots $\lambda, \overline{\lambda}$ (since K and L are real, the roots must be complex conjugate; since they are not real, they must be different: $\overline{\lambda} \neq \lambda$). The formula (3) for A and B is still valid, but it shows that A and B must also be complex conjugate:

$$\overline{A} = \overline{\left(\frac{a_0\overline{\lambda} - a_1}{\overline{\lambda} - \lambda}\right)} = \frac{a_0\lambda - a_1}{\lambda - \overline{\lambda}} = B.$$

Hence, $a_n = A\lambda^n + \overline{A}\,\overline{\lambda}^n = A\lambda^n + \overline{A\lambda^n} = 2\,\mathrm{Re}(A\lambda^n)$ (remind that the sum of a complex number z and its complex conjugate $\overline{z}$ is twice the real part of z).

Let us use for λ and A the trigonometric form:

$$\lambda = r \cdot (\cos\theta + i \sin\theta),$$
$$A = s \cdot (\cos\sigma + i \sin\sigma).$$

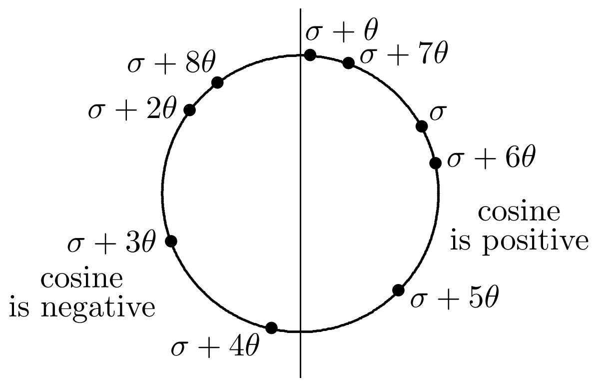
The multiplication rule for complex numbers in the trigonometric form gives:

$$A\lambda^n = sr^n(\cos(\sigma + n\theta) + i \sin(\sigma + n\theta)$$

whence

$$a_n = 2sr^n \cos(\sigma + n\theta).$$

This formula explains the sign changes in the sequence as above: since $\theta \neq 0$, the values of $\cos(\sigma + n\theta)$ switch from poisitive to negative and back (see the drawing below).



7. The case of a double root. This is the most challenging case (some mathematicians would have called it "the resonance case"). Again we begin with examples. Let $a_{n+2} = 4a_{n+1} - 4a_n$. First, let us assume that $a_0 = 0$, $a_1 = 1$. Then the sequence is

$$0, 1, 4, 12, 32, 80, 192, 448, 1024, \ldots$$

Second, let $a_0 = a_1 = 1$. Then the sequence is

$$1, 1, 0, -4, -16, -48, -128, -320, -768, \ldots$$

How to write a formula for the terms of such sequences?

The equation $\lambda^2 - 4\lambda + 4 = 0$ has only one "double" root: $\lambda = 2$. Certainly, the sequence $a_n = A \cdot 2^n$ satisfies the condition $a_{n+2} = 4a_{n+1} - 4a_n$, but this formula does not give all such sequences; in particular, neither of the two sequences given above has this form. Thus, we need to look for other (than $A \cdot 2^n$) solutions. Let us consider the problem arising in the general form.

Consider the sequences $\{a_n\}$ satisfying the condition $a_{n+2} = K \cdot a_{n+1} + L \cdot a_n$ and suppose that the polynomial $\lambda^2 - K\lambda - L$ has a double root λ, which means that $\lambda = \dfrac{K}{2}$ and $L = -\lambda^2$. Let us show that in this case not only $a_n = \lambda^n$, but also $a_n = n\lambda^n$ satisfies our condition. Indeed, our condition is $a_{n+2} = K \cdot a_{n+1} + L \cdot a_n = 2\lambda \cdot a_{n+1} - \lambda^2 a_n$, and $a_n = n\lambda^n$ satisfies this condition:

$$\begin{aligned} 2\lambda \cdot a_{n+1} - \lambda^2 \cdot a_n &= 2\lambda \cdot (n+1)\lambda^{n+1} - \lambda^2 \cdot n\lambda^n \\ &= (2(n+1)-n)\lambda^{n+2} = (n+2)\lambda^{n+2} = a_{n+2} \end{aligned}$$

This discovery provides sufficiently many solutions of our problem: we can take $a_n = (A + Bn)\lambda^n$. the coefficients A and B can we found, if we know a_0 and a_1. Indeed, $a_0 = (A + B \cdot 0)\lambda^0 = A$ and $a_1 = (A + B) \cdot \lambda$, so $A = a_0$ and $B = \frac{a_1}{\lambda} - A = \frac{a_1 - a_0\lambda}{\lambda}$.

In particular, for the first of the two sequences above, we have $A = 0, B = \frac{1}{2}$, and $a_n = \frac{1}{2}n \cdot 2^n = n \cdot 2^{n-1}$ (you can check this). For the second sequence, $A = 1$, $B = \frac{1-2}{2} = -\frac{1}{2}$ and $a_n = \left(1 - \frac{1}{2}n\right) \cdot 2^n = (2-n) \cdot 2^{n-1}$ (this also can be easily checked).[1]

8. Who is Tribonacci? No such person has ever existed. (By the way, *Fibonacci* is also not a name: it is an abbreviation of a patronimic, *Son of Bonacci*; the real name of Fibonacci was Leonardo, he was often referred to as *Leonardo di Pisa*, that is, Leonardo from Pisa.) Still many sources mention "Tribonacci numbers," meaning a sequence $\{a_n\}$ with $a_0 = a_1 = 0, a_2 = 1$ and $a_{n+3} = a_{n+2} + a_{n+1} + a_n$:

$$0, 0, 1, 1, 2, 4, 7, 13, 24, 44, 81, 149, 274, \ldots$$

[1] It is permissible to ask, how one can guess the formula $a_n = n\lambda^n$. One can say that mathematicians have a reach intuition based on their knowledge of Analysis or Differential Equations: there are similar problems in these areas which have similar solutions. But it is also possible to guess this formula without a reference to any advanced Mathematics. This is how it can be done. Suppose that our equation $\lambda^2 - K\lambda - L = 0$ has two not equal, but very close to each other solutions: λ and $\lambda + \varepsilon$ where ε is very small. Our previous considerations provide the formula

$$a_n = A\lambda^n + B(\lambda+\varepsilon)^n = A\lambda^n + B\left(\lambda^n + n\varepsilon\lambda^{n-1} + \frac{n(n-1)}{2}\varepsilon^2\lambda^{n-2} + \ldots\right)$$

If we want that the terms λ^n cancel, we can put $A = -B$, and our expression for a_n will become

$$a_n = Bn\varepsilon\lambda^{n-1} + B\frac{n(n-1)}{2}\varepsilon^2\lambda^{n-2} + \ldots$$

This is almost what we need: put $B = \frac{\lambda}{\varepsilon}$ and get

$$a_n = n\lambda^n + \frac{n(n-1)}{2}\varepsilon\lambda^{n-1} + \ldots$$

which becomes $n\lambda^n$ when ε becomes negligibly small.

Expectably, we will consider a more general construction: a sequence $\{a_n\}$ with $a_{n+3} = K \cdot a_{n+2} + L \cdot a_{n+1} + M \cdot a_{n+2}$; this sequence is uniquely determined by arbitrary chosen values of a_0, a_1, and a_2.

At the first glance, the situation with this sequence is not much different from the situation considered in previous section. We look for sequences satisfying the condition $a_{n+3} = K \cdot a_{n+2} + L \cdot a_{n+1} + M \cdot a_{n+2}$ in the form $a_n = \lambda^n$ and find that this sequence satisfies our condition if and only if λ is a solution of the equation $\lambda^3 - K\lambda^2 - L\lambda - M = 0$. In general this equation has three solutions, λ_1, λ_2, and λ_3, and if all of them are different, then all the sequences satisfying our condition are given by the formula

$$a_n = A\lambda_1^n + B\lambda_2^n + C\lambda_3^n \tag{4}$$

The coefficients A, B, and C can be expressed in terms of a_0, a_1, a_2 by means of the system of equations

$$\begin{cases} A + B + C &= a_0, \\ A\lambda_1 + B\lambda_2 + C\lambda_3 &= a_1, \\ A\lambda_1^2 + B\lambda_2^2 + C\lambda_3^2 &= a_2. \end{cases}$$

9. The solution of the cubic equation: the four cases. Speaking of the three roots $\lambda_1, \lambda_2, \lambda_3$ of the cubic polynomial $\lambda^3 - K\lambda^2 - L\lambda - M$ (with real K, L, and M), we need to distinguish four cases: (i) all of them are real and different; (ii) one root, say, λ_1 is real, and the other two are not real and complex conjugate: $\lambda_3 = \overline{\lambda}_2$; (iii) all the roots are real and $\lambda_1 \neq \lambda_2 = \lambda_3$; (iv) all the roots are real and equal to each other.

In case (i), we cannot add anything to what was said in Section 8.

In case (ii), the assumption that all a_n are real implies the complex conjugacy $C = \overline{B}$. Formula (4) takes the form

$$a_n = A\lambda_1^n + 2\,\mathrm{Re}(B\lambda_2^n) = A\lambda_1^n + 2sr^n\cos(\sigma + n\theta)$$

where

$$\lambda_2 = r(\cos\theta + i\sin\theta) \text{ and } B = s(\cos\sigma + i\sin\sigma).$$

In case (iii), formula (4) must be replaced by the formula

$$a_n = A\lambda_1^n + (B + Cn)\lambda_2^n.$$

In case (iv) we denote the only root of the polynomial by λ and formula (4) must be replaces by the formula

$$a_n = (A + Bn + Cn^2)\lambda^n.$$

The proofs of all these statements are very close to the proofs in Sections 6 and 7 (case (iv) may be regarded as new, but still it is not much different from Section 7), and we leave them to the reader.

10. Examples. The example of the Tribonacci numbers, $a_{n+3} = a_{n+2} + a_{n+1} + a_n$, $a_0 = a_1 = 0, a_2 = 1$ (see above) is not very interesting from the point of view of our general formula. The equation $\lambda^3 - \lambda^2 - \lambda - 1 = 0$ has one real and two complex conjugate solutions; the real solution is approximately 1.839287. The other two solutions have absolute values less than one, so, for large n, their contribution into a_n is negligibly small. The coefficient A is approximately 0.182804, so the approximation

$$a_n \approx 0.182804 \cdot 1.839287^n$$

is good for sufficiently large n. For example, $0.182804 \cdot 1.839287^{12} \approx 274.018$ is a good approximation for $a_{12} = 274$.

To find more appealing examples, we must consider sequences for which the cubic equation $\lambda^3 - K\lambda^2 - L\lambda - M = 0$ has better looking solutions. For example, let us consider sequences with $a_{n+3} = 2a_{n+2} + a_{n+1} - 2a_n$. The cubic equation $\lambda^3 - 2\lambda^2 - \lambda + 2 = 0$ has solutions $\lambda = 2, 1$, and -1. Thus, the sequence is described by the formula

$$a_n = A \cdot 2^n + B + C \cdot (-1)^n$$

where the coefficients A, B, C (as well as the whole sequence) are determined by a fixation of a_0, a_1, and a_2. For example, if $a_0 = a_1 = 0$ and $a_1 = 1$, then the sequence is

$$0, 0, 1, 2, 5, 10, 21, 42, \ldots$$

A computation shows that $A = \dfrac{1}{3}, B = -\dfrac{1}{2}, C = \dfrac{1}{6}$, so

$$a_n = \frac{1}{3} \cdot 2^n - \frac{1}{2} + \frac{1}{6} \cdot (-1)^n = \frac{2^{n+1} - 3 + (-1)^n}{6}$$

(you can check this, if you wish). Similarly, if we put $a_0 = 0, a_1 = a_2 = 1$, then the sequence takes the form

$$0, 1, 1, 3, 5, 11, 21, 43, \ldots,$$

and an easy computation shows that $A = \dfrac{1}{3}, C = -\dfrac{1}{3}, B = 0$. Thus, in this case

$$a_n = \frac{2^n - (-1)^n}{3}$$

(again, it can be easily checked).

Next, let us consider an example involving a double root. Let $a_{n+3} = 4a_{n+2} - 5a_{n+1} + 2a_n$. The equation $\lambda^3 - 4\lambda^2 + 5\lambda - 2 = 0$ has the solutions $2, 1, 1$. Hence, our sequence is described by the formula $a_n = A \cdot 2^n + (Bn + C)$ with A, B, and C depending on a_0, a_1, a_2. For example, if $a_0 = a_1 = 0, a_2 = 1$, then the sequence will be

$$0, 0, 1, 4, 11, 26, 57, 120, 247, \ldots,$$

and a computation shows that $A = 1, B = C = -1$. Thus, $a_n = 2^n - (n + 1)$.

Finally, let us consider an example for which our cubic equation has a triple root. Namely, let $a_{n+3} = 6a_{n+2} - 12a_{n+1} + 8a_n$. The equation $\lambda^3 - 6\lambda^2 + 12\lambda - 8 = 0$ has a triple root $\lambda = 2$. Hence, whichever a_0, a_1, a_2 we choose, the sequence is described by the formula $a_n = (A + Bn + Cn^2) \cdot 2^n$ with some A, B, C. For example, if we put $a_0 = a_1 = 0$, and $a_2 = 1$, then the sequence will be

$$0, 0, 1, 6, 24, 80, 240, 672, \ldots$$

and a computation shows that $A = \frac{1}{8}, B = -\frac{1}{8}, C = 0$, so

$$a_n = \frac{1}{8}(n^2 - n) \cdot 2^n = 2^{n-3} n(n-1).$$

A reader who likes this game can take other sets $\{a_0, a_1, a_2\}$ and obtain for sequences arising formulas like the one above. Also, the reader can consider "Tetrabonacci numbers," "Pentabonacci numbers," and subsequent families of numbers, but we stop here.