Berkeley Math Circle: Monthly Contest 8

Due May 3, 2016

Instructions

- This contest consists of seven problems of varying difficulty. Problems 1–4 comprise the *Beginner Contest* (for grades 8 and below) and Problems 3–7 comprise the *Advanced Contest* (for grades 9–12). Contest 8 is due on May 3, 2016.
- Begin each submission with your name, grade, school, BMC level, the problem number, and the contest number on every sheet. An example header:

  BMC Monthly Contest 8, Problem 3
  Bart Simpson
  Grade 5, BMC Beginner
  from Springfield Middle School, Springfield

  Submit **different problems on different pages** as they are graded separately.
- Each problem is worth seven points; to receive full points all results must be completely proven. Include all relevant explanations in words and all intermediate calculations; answers without justification will receive little or no credit. Submit solutions to as many problems as you can since partial credit will be awarded for sufficient progress.
- Remember you are not allowed to talk to anyone else about the problems, but you may consult any book you wish. See the BMC website at `http://mathcircle.berkeley.edu` for the full rules. Enjoy solving these problems and good luck!

Problems for Contest 8

1. Compute the prime factorization of $25^3 - 27^2$.

2. Suppose positive reals a, b, c, d, e satisfy

$$a + b = c^2,\ b + c = d^2,\ c + d = e^2,\ d + e = a^2,\ e + a = b^2.$$

   Prove that $a = b = c = d = e = 2$.

3. Find all real numbers $x < 5$ satisfying

$$\sqrt{5 - x} = 5 - x^2.$$

4. In parallelogram $ABCD$, E and F are points on sides AD and AB satisfying $BE = DF$. Lines BE and DF intersect at G. Show that $\angle BGC = \angle DGC$.

5. Let ABC be an acute triangle with circumcenter O and cirumcircle Ω. The tangents to Ω at B and C intersect again at point A'; points B' and C' are defined similarly. Ray AO meets segment BC at P. Prove that line $A'P$ bisects segment $B'C'$.

6. For positive real numbers a, b, c, prove that

$$\frac{1}{a+\frac{1}{b}+1}+\frac{1}{b+\frac{1}{c}+1}+\frac{1}{c+\frac{1}{a}+1} \geq \frac{3}{\sqrt[3]{abc}+\frac{1}{\sqrt[3]{abc}}+1}.$$

7. Consider an orphanage where every pair of orphans is either friends or enemies, and suppose for every three of an orphan's friends, an even number of the $\binom{3}{2}$ pairs of them are enemies. Prove that one can assign each orphan two parents such that

   (i) No pair of enemies has a common parent.

   (ii) Every pair of friends has exactly one common parent.

   (iii) There do not exist three parents for which any two parents have a common child.