Berkeley Math Circle: Monthly Contest 6 Solutions

1. A fair coin is flipped nine times. Which is more likely, having exactly four heads or having exactly five heads?

   Solution. Both outcomes are equally likely! Notice that for every sequence with four heads (such as $HTTHHTHTT$) there is a corresponding sequence with exactly five heads formed by reversing all the coin flips (in this case $THHTHTHH$). Thus the number of sequences of nine flips with exactly four heads is equal to the number of sequences with exactly five heads; since every sequence is equally likely this completes the proof.

   In fact, both probabilities are equal to $\frac{\binom{9}{4}}{2^9} = \frac{\binom{9}{5}}{2^9} = \frac{126}{512}$. □

2. Let a and b be positive real numbers. Prove that

   $$\sqrt{a^2 - ab + b^2} \geq \frac{a+b}{2}.$$

   Solution. Squaring both sides (which is OK since both sides are positive), it's equivalent to show that $4(a^2 - ab + b^2) \geq (a+b)^2$. But their difference is

   $$4(a^2 - ab + b^2) - (a+b)^2 = 3a^2 - 6ab + 3b^2 = 3(a-b)^2 \geq 0.$$

   □

3. Let A and B be two points on the plane with $AB = 7$. What is the set of points P such that $PA^2 = PB^2 - 7$?

   Solution. If we let K be the point on AB with $AK = 4$, $BK = 3$, then the answer is the line through K perpendicular to AB. To see this, set $A = (0,0)$ and $B = (7,0)$. Then the points $P = (x,y)$ are exactly those satisfying

   $$(x-0)^2 + (y-0)^2 = (x-7)^2 + (y-0)^2 - 7$$

   which rearranges to $9 = -14x + 42$, *id est* $x = 3$. □

4. The numbers 1, 2, ..., 50 are written on a blackboard. We may erase two numbers a and b, and replace both with $a + b + 2ab$; we repeat this operation until only one number remains. Prove that the value of this last number does not depend on how the operations were performed.

   Solution. Suppose at some point the numbers on the board are $x_1, \ldots, x_k$. We claim that the quantity $(2x_1+1)(2x_2+1)\ldots(2x_k+1)$ does not change. Indeed, this follows from the identity $1 + 2(a + b + 2ab) = (1 + 2a)(1 + 2b)$.

   Thus, if there is exactly one number M on the board, it is given exactly by $2M+1 = 3 \cdot 5 \cdot \cdots \cdot 101$, and in particular does not depend on the choice of operations. □

5. Show that $\sin 10^\circ$ is irrational.

Solution. One can show the triple-angle identity

$$\sin(3\theta) = 3\sin\theta - 4\sin^3\theta.$$

Thus letting $x = 2\sin(10^\circ)$ we derive

$$x^3 - 3x + 1 = 0.$$

This polynomial has no rational roots (by, say, Rational Root Theorem), hence x is irrational, whence $\sin 10^\circ$ is irrational. □

6. Let $c > 0$ be a positive real number. We define the sequence (x_n) by $x_0 = 0$ and

$$x_{n+1} = x_n^2 + c$$

for each $n \geq 0$. For which values of c is it true that $|x_n| < 2016$ for all n?

Solution. The answer is $c \leq \frac{1}{4}$.

First, we show that $c \leq \frac{1}{4}$ all work. Clearly it suffices to prove the result when $c = \frac{1}{4}$. In that case, the sequence is defined by $x_{n+1} = x_n^2 + \frac{1}{4}$. We claim that $x_n \leq \frac{1}{2}$ for all n. Indeed, this follows by induction, since it is true for $n = 1$ and for the inductive step we have

$$x_{n+1} = x_n^2 + \frac{1}{4} \leq \frac{1}{4} + \frac{1}{4} = \frac{1}{2}.$$

Now we show all $c > \frac{1}{4}$ fail. Assume on the contrary that $|x_n| < 2016$ for all n. Then since the real interval $[0, 2016]$ is compact, we must have that x_n converges to some limit x. This limit must then satisfy

$$x^2 = x + c.$$

Thus the discriminant $1 - 4c$ is nonnegative, so $c \leq \frac{1}{4}$ must hold. □

7. Let ABC be a triangle, and let X, Y, Z be the excenters opposite A, B, C. The incircle of triangle ABC touches BC, CA, AB at points D, E, F. Finally, let I and O denote the incenter and circumcenter of triangle ABC.

Prove that lines DX, EY, FZ, IO are concurrent.

Solution. The fact that DX, EY, FZ are concurrent follows from the fact that triangles DEF and XYZ are homothetic; indeed, note that EF and YZ are both perpendicular to the internal angle bisector of $\angle BAC$.

Now, to see that the concurrence point lies on IO, note that point I the orthocenter of triangle XYZ, and O is the nine-point center of triangle XYZ. Thus line IO is the Euler line of triangle XYZ and thus passes through the circumcenter S of triangle XYZ. But I is the circumcenter of triangle DEF, hence line SI passes through the concurrency point. □